Uniform Weighting in 2D Fan Beam Reconstruction

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I. BACKGROUND

An elementary acquisition configuration for 2D transmission tomography is the (ideal) fan beam geometry, where the X-ray source moves on a circular trajectory $\underline{a}(\lambda) = R(\cos\lambda, \sin\lambda)$ around the object of interest $f(\underline{x})$ [1]. An appropriately posed, e.g. curved detector helps to acquire

$$g(\lambda,\gamma) = \int_0^{+\infty} dt\, f\left(\underline{a}(\lambda) - t\cos\gamma \cdot \underline{e}_w(\lambda) + t\sin\gamma \cdot \underline{e}_u(\lambda)\right) \qquad (1)$$

with $\underline{e}_w(\lambda) = (\cos\lambda, \sin\lambda)$ and $\underline{e}_u(\lambda) = (-\sin\lambda, \cos\lambda)$ and by that measures 2D Radon data $\mathcal{R}f$ of the object.

The theoretically minimal set of projection data for an exact reconstruction of $f(\underline{x})$ is obtained with the so called short scan configuration that usually leads to a certain amount of redundancy. Some Radon values are measured twice, since $g(\lambda,\gamma) = g(\lambda+\pi-2\gamma, -\gamma)$ - others only once, as visualized in Fig. 1. In FBP reconstruction, one needs to equalize the contribution of every Radon data to avoid severe gradient artefacts in the final image. A common approach is to multiply $g(\lambda,\gamma)$ with a weighting function $w(\lambda,\gamma)$, where $w(\lambda,\gamma) + w(\lambda+\pi-2\gamma,-\gamma) = 1$. In FBP algorithms based on the conventional 2D Radon inversion formula [1], the weighting is followed by a high-pass filtering step, so that only smooth functions $w(\lambda,\gamma)$ can be considered. In general, however, the non-smooth uniform weighting with $w = w_{ns} = \{0.5, 1\}$ for redundant or non-redundant values, resp., is preferred. Especially for real world data, that suffers from noise effects, superior SNR is expected in the reconstructed $f(\underline{x})$.

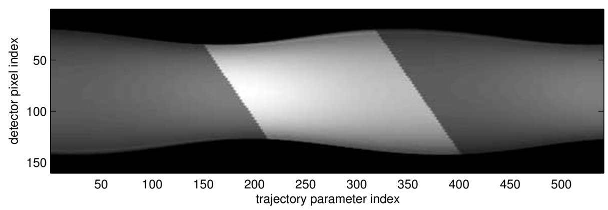

Fig. 1: Fan-beam data of the Shepp Logan phantom for $\lambda \in [0°, 270°]$. The darker areas correspond to redundant line integral data.

II. METHODS

The smoothness restriction for w is theoretically overcome with the introduction of a novel 2D FBP formula [2], where the weighting can be applied after filtering. Here, an accurate approach for reconstruction with uniform weighting is presented using this novel formula. Consider a point $\underline{x}$. From [2], we can write

$$f(\underline{x}) = \frac{1}{2\pi}\int_\Lambda d\lambda \frac{1}{\|\underline{x} - \underline{a}(\lambda)\|} w_{ns}(\lambda, \underline{x}) g_F(\lambda, \gamma(\underline{x})) \qquad (2)$$

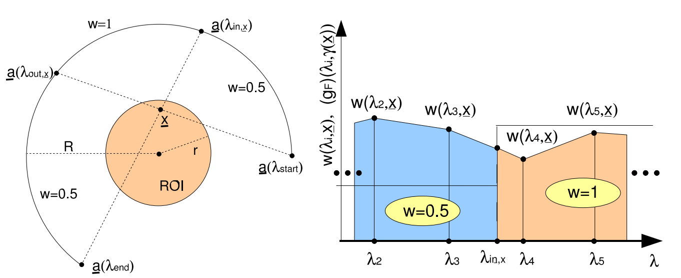

Fig. 2: The two lines through $\underline{x}$ and the endpoints of $\underline{a}(\lambda)$ define the steps in the weighting function. A valid application of the trapezoidal rule requires a special handling for e.g. $[\lambda_3, \lambda_4]$.

where $g_F(\lambda,\gamma)$ is some appropriately filtered projection data and $\gamma(\underline{x})$ denotes the ray through $\underline{x}$.

Using basic geometric considerations (Fig. 2) we compute $\lambda_{in,\underline{x}}$ and $\lambda_{out,\underline{x}}$, that coincide with the discontinuities of the weighting function. Radon data for the line through $\underline{x}$ and $\underline{a}(\lambda), \lambda \in [\lambda_{in,\underline{x}}, \lambda_{out,\underline{x}}]$ is non-redundant ($w_{ns} = 1$), whereas Radon data corresponding to lines at $\lambda \notin [\lambda_{in,\underline{x}}, \lambda_{out,\underline{x}}]$ occurs twice, hence $w_{ns} = 0.5$. This concept is called *pixel based weighting*. To handle the discrete character of practical data, the integral in 2 is approximated with the trapezoidal rule. Fig. 2 illustrates an interval of $g_F(\lambda, \gamma(\underline{x}))$ and the corresponding $w(\lambda, \underline{x})$. Whereas the area of most trapezoids contribute with a weight 0.5 or 1 to the total sum, the trapezoid $T_{[\lambda_3,\lambda_4]}$ includes the step in w at $\lambda_{in,\underline{x}}$. Its area can be split into two parts $A_{[\lambda_3,\lambda_{in,\underline{x}}]}$ and $A_{[\lambda_{in,\underline{x}},\lambda_4]}$, yielding its total contribution

$$\begin{aligned}&0.5 \cdot A_{[\lambda_3,\lambda_{in,\underline{x}}]} + 1.0 \cdot A_{[\lambda_{in,\underline{x}},\lambda_4]} = \\ &\left(\frac{\Delta\lambda - d}{2} + \frac{d^2}{4\Delta\lambda}\right) g_F(\lambda_3, \gamma(\underline{x})) + \left(\frac{\Delta\lambda}{2} - \frac{d^2}{4\Delta\lambda}\right) g_F(\lambda_4, \gamma(\underline{x})),\end{aligned} \qquad (3)$$

with $d = \lambda_{in,\underline{x}} - \lambda_3$. Similar computations are done for $\lambda_{out,\underline{x}}$ and reordering the summands yields a valid weight $w \in [0.5, 1]$ for every $g_F(\lambda, \gamma(\underline{x}))$.

III. RESULTS

To evaluate the SNR performance of pixel based weighting, sinogram data from a water cylinder ($r = 9$cm) is simulated ($\Delta\lambda = 0.4°$, $\lambda \in [0°, 330°]$, $R = 75$cm) with added Poisson noise. Fig. 3 presents a comparison between the generalized Parker weight [3, 4] and the pixel based approach, showing the superiority of the latter in terms of average noise and equal noise distribution.

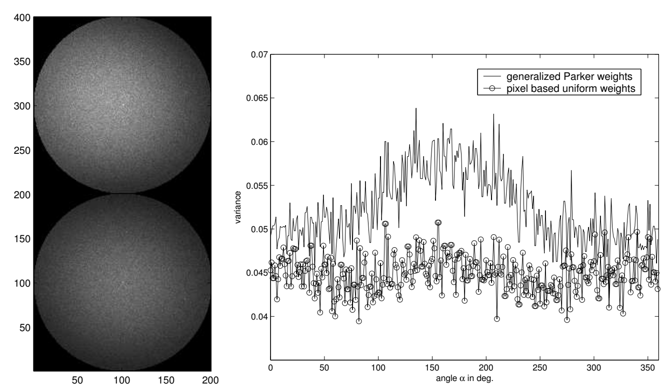

Fig. 3: Left: Variance images from 250 reconstructions: (top) Parker weight, (bottom) pixel based weight. Right: variance plot at $\underline{x} = (a\cos\alpha, a\sin\alpha)$ with $a = 5$cm.

IV. DISCUSSION

First results show that pixel based weighting can improve the SNR in reconstructions from redundant data compared to common methods, due to a more balanced and exact use of redundancies. Future work may investigate other pixel based weighting schemes (e.g. to optimize spatial resolution instead of SNR). The transfer of the introduced concepts into 3D is considered to be of major interest.

REFERENCES

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