Factorization of the Circular Cone-Beam Reconstruction Problem

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I. Introduction

Image reconstruction from cone-beam (CB) data acquired on a circular trajectory is an active subject of research nowadays. Unfortunately, circular data acquisition does not provide a complete set of 3D Radon data and the amount of missing information increases when considering the practically relevant short-scan trajectory rather than a full-scan. Accurate reconstruction of an image volume is therefore in general impossible. Over the years, several (approximate) CB reconstruction techniques have been developed for short-scan circular trajectories, but no existing technique lies on a firm theoretical ground while allowing some degree of data truncation. The existence of a firm theory supporting the algorithm, however, is important to be able to predict artifacts and to analyze numerical stability. In the following, we introduce such a theory that comes along with a factorization of the CB reconstruction problem into a set of independent 2D inversion problems. The factorization approach is based on (i) the general and exact CB reconstruction theory derived in [1] and on (ii) the principles of a novel, heuristically-derived CB reconstruction approach recently published by Yu *et al.* [2]. As such, our theory allows partial transverse truncation for specific regions-of-interest (ROIs).

II. Reconstruction Theory

The derivations assume a circular source trajectory, which is parametrized by λ. For the considered short scan, the source covers an angular interval $[0, \lambda_m]$ with $\lambda_m < 2\pi$. Let us first define a family of planes that intersect the object ROI and are orthogonal to the plane of the source trajectory. Without losing generality, these planes are selected to contain the source point at $\lambda = 0$. A certain element of that family may be uniquely specified by an angular parameter ϕ. Each so called ϕ-plane has a second intersection with the source trajectory at $\lambda_2(\phi) < \lambda_m$. See Fig. 1 for an illustration of the considered set-up. For every ϕ-plane, two Cartesian coordinates t and z can be introduced, with z in the direction orthogonal to the plane of the circular scan. A point within the 3D ROI may then be parametrized using the non-Cartesian coordinates $\underline{x}(\phi, t, z)$. We consider a reconstruction

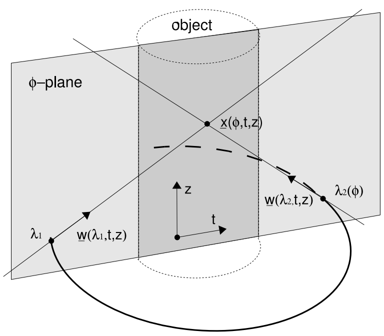

Fig. 1. Illustration of the proposed factorization of CB reconstruction.

method that consists of two steps. First, CB data is differentiated with respect to the horizontal detector coordinate and backprojected onto the (ϕ, t, z) sampling grid of the ROI. For each ϕ-plane, only CB data within the interval $[0, \lambda_2(\phi)]$ is backprojected. For the boundaries of that interval, an additional backprojection term needs to be considered. This yields an intermediate function $b(\phi, t, z)$ that is related according to [1] to the sought object function f by the equation

$$\begin{aligned} \frac{1}{\pi} b(\phi, t, z) &= \int_{-\infty}^{\infty} \frac{dt}{\pi t} f\left(\underline{x}(\phi, t, z) + t\underline{w}_1(\phi, t, z)\right) \\ &- \int_{-\infty}^{\infty} \frac{dt}{\pi t} f\left(\underline{x}(\phi, t, z) + t\underline{w}_2(\phi, t, z)\right), \quad (1) \end{aligned}$$

where $\underline{w}_1(\phi, t, z)$ (resp. $\underline{w}_2(\phi, t, z)$) is the unit vector from the source position at $\lambda_1(\phi)$ (resp. $\lambda_2(\phi)$) to $\underline{x}(\phi, t, z)$.

The backprojection result $b(\phi, t, z)$ on a fixed ϕ-plane is related only to values of f within this plane and this fact holds for every ϕ. By that, we were able to reduce the CB reconstruction problem in a set of independent 2D inversion problems, each of which corresponds to finding the density values f in one ϕ-plane from the function b according to relation (1).

For each ϕ-plane, a linear system of equations $A\underline{f} = \underline{b}$ can be determined, which models relation (1). The vectors $\underline{f}$ and $\underline{b}$ define samples of f and b in that plane. Then, reconstruction is reduced to finding a solution $\underline{f}$ to the system of equations and a detailed SVD analysis of variations of the reconstruction problem becomes feasible. A nice property of that modeling is that the linear system can easily be extended by adding additional knowledge, such as the integrals of the values of f along the lines of direction $\underline{w}_1$ and $\underline{w}_2$ through $\underline{x}(\phi, t, z)$, which are part of the measured CB data. Also, ray integrals as obtained from source positions corresponding to an additional scan segment can be used to extend the system matrix A.

III. Numerical Results and Discussion

The presented theory is now used to study (i) the influence of the scan radius R and (ii) the influence of an additional line scan on the stability of the CB reconstruction problem. The evaluation assumes a short object so that CB data acquired with a vertically centered, flat panel detector of height $d = 50cm$ is free of axial truncation. The cylindrical ROI has radius $r = 15$cm and the sampling of a ϕ-plane uses the pixel spacing $\Delta t = \Delta z = 5$ mm. For each investigated reconstruction scenario, we consider the ϕ-plane which goes through the rotational axis and visualize the spectrum of singular values σ_i of the corresponding matrix A. Since contributions corresponding to small singular values can be recovered less accurately, the spectra allow to deduce the numerical stability of the reconstruction of f within the ϕ-plane. From results presented in the left diagram of Fig. 2, we notice that the curves obtained for small R decrease faster compared to the spectra obtained with larger radii. That indicates a "proportional" relation between R and the numerical stability. In a second experiment, the radius is fixed to $R = 700$ mm, but an additional vertical line scan is attached to the circular trajectory at $\lambda = 0$, so that the line belongs to each ϕ-plane. Along the line scan, a variable number of 1 to 9 additional vertex points (VPs) are equidistantly distributed in the height interval $[0, H]$ mm and from every VP, a large fan of rays emerges, covering the complete height of the object. The additional integral information helps to stabilize the inversion problem, as illustrated in the right diagram of Fig. 2, since an increasing number of VPs elevates the corresponding singular value spectra.

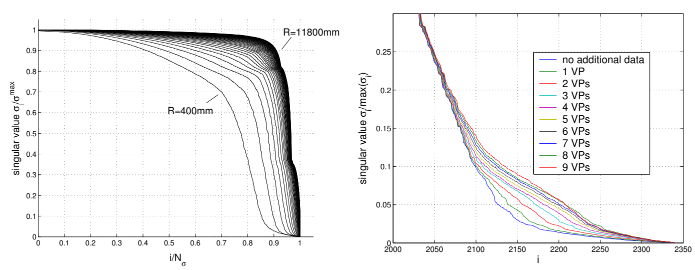

Fig. 2. Normalized singular value spectra $\sigma_i / \max(\sigma_i)$ of various acquisition set-ups. (left) Spectra for $R \in [400, 11800]$ mm and $\Delta R = 300$ mm. For increasing R, the curves enclose a larger area with the coordinate axes. (right) Spectra for $R = 700$ mm with additional integral data corresponding to VPs distributed along a vertical line scan. Only the smallest σ_i are presented.

IV. Conclusions

We introduced a novel factorization of the reconstruction problem in circular CB tomography into a set of independent 2D inversion problems. These inversion problems can be represented by a linear system of equations. Therefore, a detailed and quantitative analysis of the numerical stability of variations of the reconstruction scenario becomes possible. We obtained first results that are consistent with other data sufficiency theories, established e.g. in the context of the 3D Radon inversion theory.

References

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[2] L. Yu, D. Xia, Y. Zou, X. Pan, C. A. Pelizzari, and P. Munro. Image reconstruction from truncated data in circular cone-beam CT and its applications to radiotherapy. In *The 8th intern. meeting on fully 3D image reconstruction in radiology and nuclear medicine*, pages 391–395, 2005.