Estimation of Mean Noise Propagation in SENSE for Generation of Optimized Undersampling Patterns

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INTRODUCTION: Parallel imaging techniques like SENSE or GRAPPA are used routinely in clinical practice today and offer robust acceleration factors in the range of R=2 to R=4. The acceleration comes at the expense of local noise amplification, known as g-factor noise, which depends on the coil geometry and the undersampling pattern. Several methods have been proposed to calculate g-factor maps for visualizing the strength of the local noise amplification [1]. Recently, novel iterative reconstruction techniques have been presented that promise high acceleration factors by sampling k-space incoherently, e.g. using Poisson disc schemes, and exploiting sparse representations of the object [2]. These methods can be combined with parallel imaging techniques for even higher acceleration while offering certain degrees of freedom in the trajectory design. Here, we derive a novel measure that enables assessment of the mean g-factor noise propagation for any given sampling pattern. The measure can be evaluated at low computational costs and hence can be used to automatically select a sampling pattern that minimizes noise amplification.

THEORY: In the conventional SENSE method the desired magnetization **M**, representing the final image, is calculated with $\mathbf{M} = [\mathbf{C}^H\mathbf{P}^H\mathbf{\Psi}^{-1}\mathbf{PC}]^{-1}\mathbf{C}^H\mathbf{P}\mathbf{\Psi}^{-1}\mathbf{S}$, where **C** denotes the vector of coil sensitivities, **P** the projection onto the undersampling pattern defined in k-space, **Ψ** the covariance matrix between different coils and **S** the vector of image data from the different coils. Consequently, the covariance matrix of the magnetization is given by $\mathit{Cov}[\mathbf{M}] = [\mathbf{C}^H\mathbf{P}^H\mathbf{\Psi}^{-1}\mathbf{PC}]^{-1}$. This matrix inversion is computationally very expensive and it should be noted that the g-factor is computed only from the diagonal to give a measure of the noise enhancement per pixel. Here, however, we limit ourself to even less: the trace of the covariance matrix $\boldsymbol{\sigma}^2 \equiv \mathrm{Tr}[[\mathbf{C}^H\mathbf{P}^H\mathbf{\Psi}^{-1}\mathbf{PC}]^{-1}]$, which can be understood as the sum of the variance of each image pixel. It should also be noted that the expression amounts to the summed squared g-factor for decorrelated data with coil sensitivities normalized to one in each pixel.

The direct evaluation of this expression is computationally demanding [3]. Since the trace operation is invariant under unitary transformations, we transform to k-space and find $[\mathbf{C}^H\mathbf{P}^H\mathbf{\Psi}^{-1}\mathbf{PC}](\mathbf{k},\mathbf{q}) = \Sigma_n\, \Delta(\mathbf{p}_n\text{-}\mathbf{k},\mathbf{p}_n\text{-}\mathbf{q})$. Here the sum runs over all sampled k-space indices $\{\mathbf{p}_n\}$ and we introduced $\Delta(\mathbf{p},\mathbf{q}) = \Sigma_{I,J}\, \mathbf{c}_I^*(\mathbf{p})\, \Psi^{-1}_{IJ}\, \mathbf{c}_J(\mathbf{q})$ with coil sensitivities in k-space labelled as $\mathbf{c}_I$.

The crucial observation is then to look at the properties of the positive, hermitian matrix **Δ**: assuming smoothly varying coil profiles the entries strongly peak with both indices **p**,**q** close to k-space center. The diagonal entries are sums of positive numbers, whereas off-diagonal entries potentially show cancellations through interference effects. We therefore make the ad-hoc approximation $\Delta(\mathbf{p},\mathbf{q}) \approx \Delta(\mathbf{p})\, \delta_{\mathbf{p},\mathbf{q}}$, where we use a Konecker delta and $\Delta(\mathbf{p}) \equiv \Delta(\mathbf{p},\mathbf{p})$. With this approximation we finally obtain an estimate of the mean noise propagation

$$\sigma_{\mathrm{estimate}}{}^2 / \mathbf{N}_{\mathrm{pixels}} = \sum\nolimits_{\mathbf{k}} \frac{1}{\sum_{\mathbf{p}_n} \Delta(\mathbf{p}_n - \mathbf{k})} \qquad (1)$$

for the sum of each pixel's variance divided by the number of pixels. This expression can be evaluated efficiently, especially when the support of coil sensitivities in k-space is small. Another way to look at our approximation in Eq.(1) is the observation that the best result for $\sigma_{\mathrm{estimate}}{}^2/\mathbf{N}_{\mathrm{pixels}}$ is obtained for a diagonal of $\Delta(\mathbf{p},\mathbf{q})$ with almost equal entries. In this light trying to minimize $\sigma_{\mathrm{estimate}}{}^2/\mathbf{N}_{\mathrm{pixels}}$ corresponds to making the diagonal of $[\mathbf{C}^H\mathbf{P}^H\mathbf{\Psi}^{-1}\mathbf{PC}](\mathbf{k},\mathbf{q})$ as equally distributed as possible.

MATERIALS AND METHODS: Phantom measurements were performed with a 3D FLASH sequence on a 3T MR system (MAGNETOM Verio, Siemens AG, Erlangen, Germany) using 12 and 32-channel head coils and a 30-channel body-matrix/spine coil array. Coil sensitivities were estimated using the algorithm proposed in [4], after noise decorrelation of the data. Images were calculated using the CG-SENSE algorithm [5].

	32-channel head coil ; 30-channel body-matrix/spine array							
R_{PE}	1	2	1	3	1	4	2	1
R_{PAR}	1	1	2	1	3	1	2	4
$\sigma_{estimate}^2/N_{pixels}$	1;1	2.00;2.25	2.41;2.27	3.45;3.96	4.80;4.37	5.45;6.23	4.82;4.97	7.39;6.62
σ^2/N_{pixels}	1;1	1.06;1.53	1.71;1.60	1.62;20.3	11.7;270.	3.90;1157.	1.84;5.54	49.5;2814.

Tab. 1: *Evaluation for two different datasets (black and blue) for regular SENSE pattern with acceleration factor* R_{PE} *in phase-encoding and* R_{PAR} *in partition direction.*

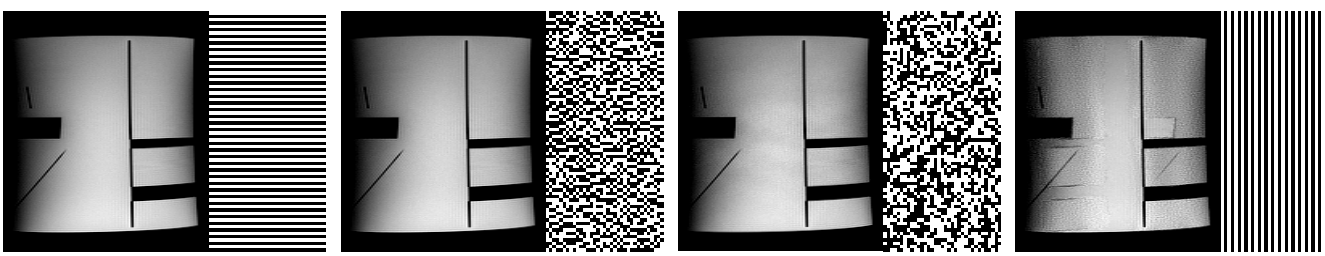

Fig. 1: *Images reconstructed from a 12-channel head coil dataset for a total acceleration of **R**=2 with an exemplary fraction of the undersampling pattern on the right. Incoherent patterns are generated with a deformed Poisson-disc algorithm. From left to right:* $\sigma_{estimate}^2/N_{pixels}$*=2.10, 2.18, 2.29, 3.15 and for the* L_2*-norm of the difference to the fully sampled image by its norm* $\|\Delta I\|_2/\|I_{full}\|_2$*=0.028, 0.031, 0.126, 0.666.*

RESULTS AND DISCUSSION: In order to test and validate $\sigma_{estimate}^2/N_{pixels}$ as noise measure, we retrospectively generated undersampled datasets from fully sampled data using several regular SENSE patterns with accelerations in 2D. $\sigma_{estimate}^2/N_{pixels}$ as well as σ^2/N_{pixels} (corresponding to the mean squared g-factor) were determined for these patterns. Since the SENSE reconstruction is separable in the readout direction, only the middle slices were processed for simplicity. The results for two different datasets are presented in Tab.1. Albeit the values for $\sigma_{estimate}^2/N_{pixels}$ and σ^2/N_{pixels} deviate quantitatively, especially for higher acceleration factors, the ordering remains consistent between patterns with identical total acceleration. In that sense $\sigma_{estimate}^2/N_{pixels}$ can be used for qualitative comparison of different sampling patterns with equal number of sample points.

Fig. 1 shows images reconstructed from a dataset acquired with a 12-channel head coil with total acceleration factor of R=2. In addition to the SENSE patterns, incoherent sampling patterns were used for the reconstruction. As for the SENSE patterns discussed before, there is a strong correlation between $\sigma_{estimate}^2/N_{pixels}$ and the L_2-norm of the residual between the reconstructed and fully sampled image.

CONCLUSION: This work presents a qualitative measure $\sigma_{estimate}^2/N_{pixels}$ for comparison of the noise enhancement from different undersampling patterns with equal sample count, which can be evaluated with low numerical complexity. It can be useful for automatic selection of optimal undersampling patterns for routine parallel imaging applications. Furthermore, it can be used during the iterative generation of incoherent sampling patterns with reduced noise amplification by selecting sample positions that give a low value of $\sigma_{estimate}^2/N_{pixels}$.

REFERENCES: [1] K. Pruessmann et al, MRM 1999; 42:952-962. [2] M. Lustig et al, MRM 2007; 58:1182-1195. [3] B. Liu et al, Proc. ISMRM 2008, p.1285. [4] M. Griswold et al, Proc. ISMRM 2002, p.2410. [5] K. Pruessmann et al, MRM 2001; 46:636-651.