Image representation using mollified pixels for iterative reconstruction in x-ray CT

F. Noo, *Member, IEEE,*, K. Schmitt, *Student Member, IEEE,* K. Stierstorfer, and H. Schöndube

***Abstract*—We introduce a new basis function, the mollified pixel, for image representation in x-ray computed tomography and other imaging modalities. The mollified pixel is defined as the convolution of a classical pixel with a smooth function that approximates the Dirac impulse. Since such a smooth function can be defined in an infinite number of ways, the mollified pixel defines a wide class of methods for image representation. For utilization of the mollified pixel in iterative reconstruction, it is needed to know the Radon transform of a mollified pixel. Using a link between classical pixels and B-splines, we suggest a computationally-efficient expression for this Radon transform. Preliminary iterative CT reconstruction results based on mollified pixels are provided along with our theoretical developments to demonstrate the potential benefits of mollified pixels.**

I. Introduction

Iterative reconstruction methods require a formalism that links the measurements to the desired reconstruction, f. One formalism that is often used is the basis function approach where f is approximated by a linear combination of basis functions. When adopting this approach, it is common to chose the basis functions as translated and dilated versions of a mother function. Several mother functions have been proposed in the literature; the most popular ones are the classical pixel, the B-splines [1] and the blobs [2]. The classical pixel is advantageous in terms of computational effort but yields an image representation where high frequencies are barely apodized. This feature leads to reconstructions that include significant aliasing errors, especially in comparison with using B-splines or blobs. These other basis functions yield lower errors because they provide a smoother image representation, but the gain comes with an increase in computational effort that is primarily due to the support of the mother function being twice to four times wider (depending on the basis function). In this paper, we introduce the concept of mollified pixels for iterative reconstruction. The mollified pixel is a mother function that can be as smooth as blobs or B-splines but with a smaller support.

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Frédéric Noo is with the Department of Radiology, the University of Utah, Salt Lake City, USA (e-mail: noo@ucair.med.utah.edu).

Katharina Schmitt is a Ph.D candidate with the Chair of Pattern Recognition Lab, Friedrich-Alexander-University, Erlangen, Germany (e-mail: katharina.schmitt@studium.uni-erlangen.de). She is also with the Department of Radiology, the University of Utah, Salt Lake City, USA and with Siemens AG, Healthcare Sector, Forchheim, Germany.

Joachim Hornegger is with the Chair of Pattern Recognition, Friedrich-Alexander-University of Erlangen-Nuremberg, Erlangen, Germany.

Karl Stierstorfer and Harald Schöndube are with Siemens AG, Healthcare Sector, Forchheim, Germany.

II. Classical pixels

The classical pixel representation of f corresponds to approximating f by

$$f_a = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{kl}\, b_\Delta(x - x_k, y - y_l) \tag{1}$$

where $x_k = k\,\Delta$, $y_l = l\,\Delta$ and $b_\Delta(x,y) = b(x/\Delta, y/\Delta)$ with

$$b(x,y) = \begin{cases} 1 & \text{if } |x| < 1 \text{ and } |y| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In this representation, b_Δ is a square pixel of width Δ. Moreover, the summations in k and l are usually finite since the object f is typically compactly supported.

III. Mollified pixels

Let $\Phi(x,y)$ be a smooth function that is even in both x and y, compactly-supported, and such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x,y)\,dx\,dy = 1\,. \tag{3}$$

The mollified pixel, denoted as $\hat{b}$, is defined as the convolution of Φ and b:

$$\hat{b}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x-u, y-v)\, b(u,v)\,du\,dv\,. \tag{4}$$

This pixel is used to approximate f in the same way as b was, namely f is approximated by

$$\hat{f}_a = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{kl}\, \hat{b}_\Delta(x - x_k, y - y_l) \tag{5}$$

with $\hat{b}_\Delta(x,y) = \hat{b}(x/\Delta, y/\Delta)$. However, there is a fundamental difference between (1) and (5): the representation using the mollified pixel is as smooth as the function Φ, whereas the representation using the classical pixel is piecewise constant and thus not even continuous.

IV. Radon transform of mollified pixels

To develop an iterative reconstruction method, we need a means to compute the Radon transform of $\hat{f}_a$. Using the properties of this transform, it is easily seen that

$$(\mathcal{R}\hat{f}_a)(\theta, s) = \Delta \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{kl}\, \hat{r}(\theta, s^*(k,l,\theta,s)) \tag{6}$$

where $\hat{r}(\theta, s)$ is the Radon transform of $\hat{b}$ and

$$s^*(k,l,\theta,s) = (s - x_k\,\cos\theta - y_l\,\sin\theta)/\Delta\,. \tag{7}$$

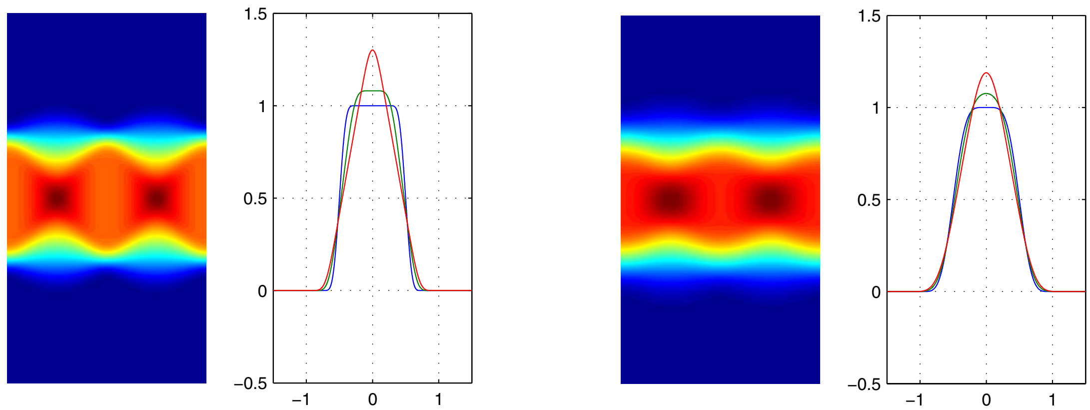

Fig. 1. Sinogram of the mollified pixel, together with plots at angles $\theta = 0$ (blue curve), $\theta = \pi/8$ (green curve) and $\theta = \pi/4$ (red curve). Left: $m = 3$ and $a = 0.25$. Right: $m = 5$ and $a = 0.5$.

Hence, the calculation of the Radon transform of $\hat{f}_a$ reduces to the calculation of that of $\hat{b}$, just like for blobs and B-splines.

To obtain the Radon transform of $\hat{b}$, we again use a property of the Radon transform:

$$\hat{r}(\theta, s) = \int_{-\infty}^{\infty} (\mathcal{R}\Phi)(\theta, s - s')\, r(\theta, s')\, ds' , \tag{8}$$

where r and $\mathcal{R}\Phi$ are the Radon transforms of b and Φ, respectively. At this stage, we recall that r can be written in the following form [1]:

$$r(\theta, s) = \frac{1}{\cos\theta\, \sin\theta} \sum_{m=0}^{1} \sum_{n=0}^{1} (-1)^{k+l}\, \gamma\left(s + d(k, l, \theta)\right) , \tag{9}$$

where $d(k, l, \theta) = (1/2 - k)\, \cos\theta + (1/2 - l)\, \sin\theta$ and

$$\gamma(t) = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases} . \tag{10}$$

Hence,

$$\hat{r}(\theta, s) = \frac{1}{\cos\theta\, \sin\theta} \sum_{m=0}^{1} \sum_{n=0}^{1} (-1)^{k+l}\, \hat{\gamma}_\theta\left(s + d(k, l, \theta)\right) \tag{11}$$

where

$$\hat{\gamma}_\theta(t) = \int_{-\infty}^{\infty} (\mathcal{R}\Phi)(\theta, s - s')\, \gamma(s')\, ds' . \tag{12}$$

This last form is adopted for our computations.

V. Selection of the Mollifier

We do not define Φ explicitly because we are primarily interested in $\hat{\gamma}_\theta$ and few expressions for Φ allow an analytical calculation of $\hat{\gamma}_\theta$. Moreover, the result of this calculation could easily be an unwieldy expression.

Instead, we note from (12) that, for each θ, $\hat{\gamma}_\theta$ can be directly defined as a mollified version of γ. Moreover, the dependence of $\hat{\gamma}_\theta$ on θ can be omitted by considering the same mollifying operation for all θ, which is equivalent to indirectly using a function Φ that is circularly-symmetric. To implement this approach, observe that

$$\frac{d^2\gamma}{dt^2} = \delta(t) , \tag{13}$$

where $\delta(t)$ is the Dirac impulse. This equation shows that the second derivative of $\hat{\gamma}_\theta$ can be chosen as any mollifier that approximates $\delta(t)$. One example is

$$\frac{d^2\hat{\gamma}_\theta}{dt^2} = \begin{cases} \alpha_m \left[1 - \left(\frac{t}{a}\right)^2\right]^m & \text{if } |t| < a \\ 0 & \text{otherwise} \end{cases} . \tag{14}$$

In this expression, m and a control the smoothness and the width of the mollifier, respectively, whereas α_m is such that the right hand side in (14) integrates to one.

The expression to obtain $\hat{\gamma}_\theta$ from its second derivative is

$$\hat{\gamma}_\theta(t) = \int_{-a}^{t} (t - u)\, \frac{d^2\hat{\gamma}_\theta}{du^2}\, du . \tag{15}$$

Fig. 1 illustrates the behavior of $\hat{r}$ for two different selections of m and a.

Note that our approach does not imply that Φ cannot be obtained: Φ can always be numerically computed from its Radon transform using the following equation:

$$(\mathcal{R}\Phi)(\theta, s) = \frac{d^2\hat{\gamma}_\theta}{ds^2} \tag{16}$$

VI. Application

The image representation using mollified pixels was tested for iterative reconstruction from CT measurements in parallel-beam geometry. The objective function to be minimized was defined as the least-square difference between the measurements and $\mathcal{R}\hat{f}_a$, which corresponds to seeking the maximum-likelihood solution under the assumption of independent normally-distributed measurements with stationary variance. This assumption is satisfactory for brain CT scans performed with tube current modulation and a properly-shaped bowtie filter. The test was performed using the FORBILD head phantom with no ears, using square pixels of width 0.75 mm, and using a sinogram consisting of 800 views with a ray width of 0.75 mm. The results displayed in Fig. 2 illustrate the impact of the shape of the pixel on discretization errors. These results were obtained from 200 iterations of the Landweber algorithm, using a regularization factor that was very close to the maximum value allowed. Note that the over- and

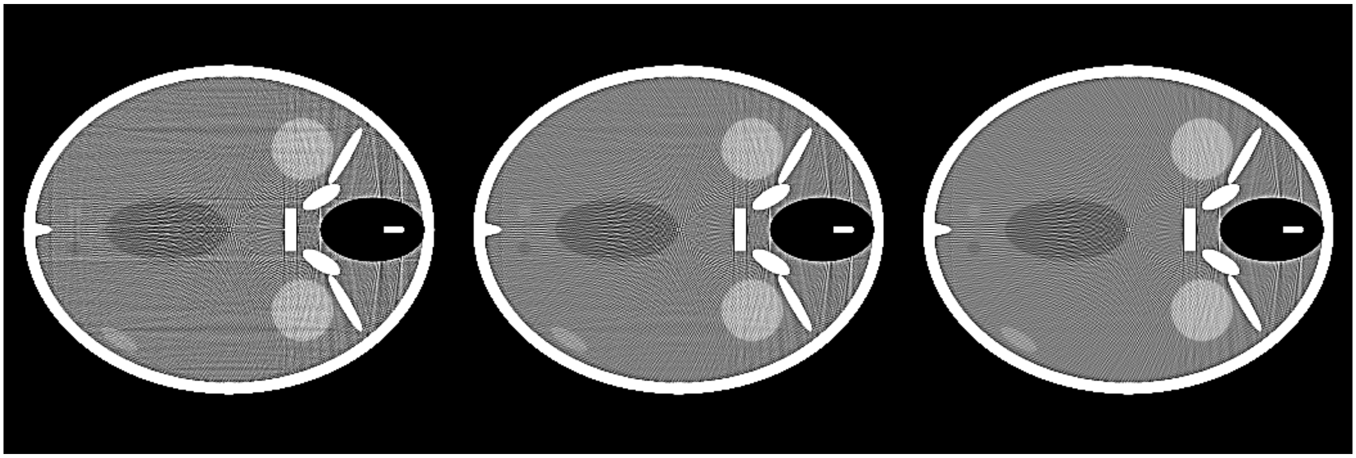

Fig. 2. Iterative reconstructions with (left) classical pixels, (middle) mollified pixels ($m = 3$, $a = 0.25$), (right) mollified ($m = 5$, $a = 0.5$), respectively. Grayscale (c/w): 50/50 HU

undershoots near the high-discontinuities of the phantom are typical of iterative reconstructions with no penalty terms, irrespective of the forward projection model.

VII. Discussion

We have introduced the concept of mollified pixels for iterative reconstruction. These pixels present two significant advantages over blobs: they can be more compact at a same degree of smoothness, and they partition unity, so that constant functions can be exactly represented. In comparison with the B-splines, they are more compact and the computation of their Radon transform can be performed more efficiently. From an implementation viewpoint, we found that a tabulation of the values of $\hat{\gamma}_\theta$ over the interval $|t| < a$ is simple and provides a significant gain in efficiency.

Comparison with methods that do not rely on basis functions such as that reported in [3] would also be of interest, however we note that the basis-function approach offers the advantage of a straight separation between data model and image representation, and the approach in [3] is limited in terms of smoothness in the image representation. Finally, although the results were presented in 2D, we expect that mollified voxels can be defined using a similar machinery for 3D reconstruction. Extension to 3D reconstruction is the subject of future investigations.

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