Analog to Digital Conversion: Quantization

from the Perspective of Pattern Recognition

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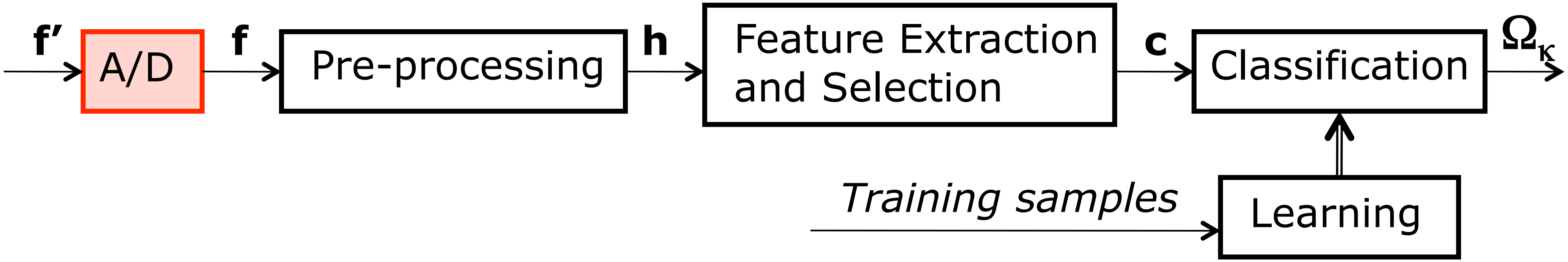
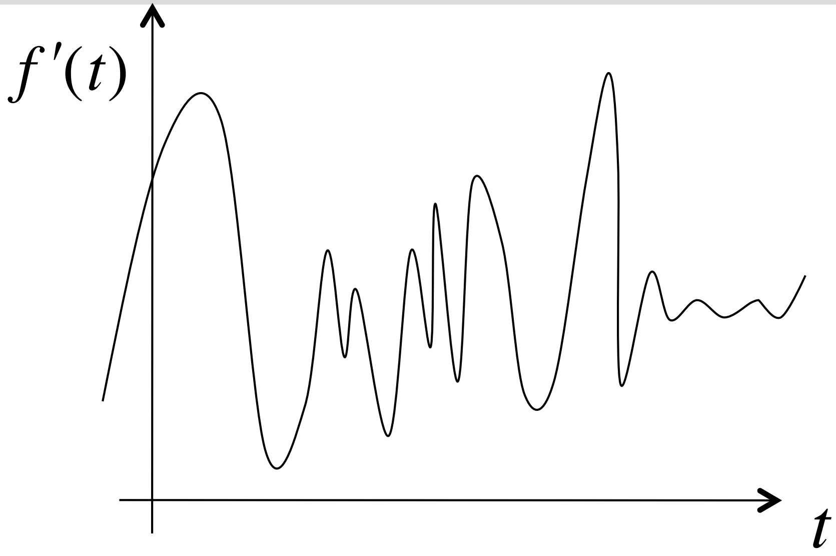
Pattern Recognition Pipeline



- The goal of analog to digital conversion is to gather sensed data f' and change it to a representation that is amenable to further digital processing.

Need for A/D Conversion



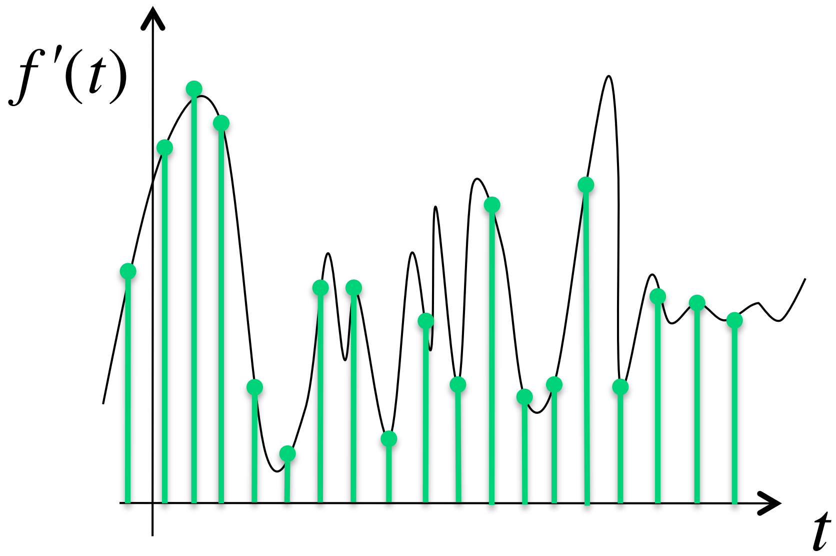
- Continuous range of t values
- Continuous range of amplitude $f'(t)$ values.
- We can only store a finite amount of values
- in a finite number of bits (discrete values).
- Goal: Find a discrete representation such that the original analog signal can be accurately reconstructed.

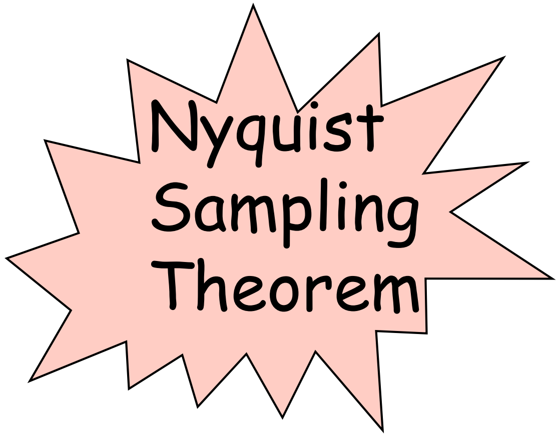
A/D Conversion Steps

- The A/D conversion (coding) involves:

1. measuring the amplitude values (or function values) at a finite number of positions:

   sampling,

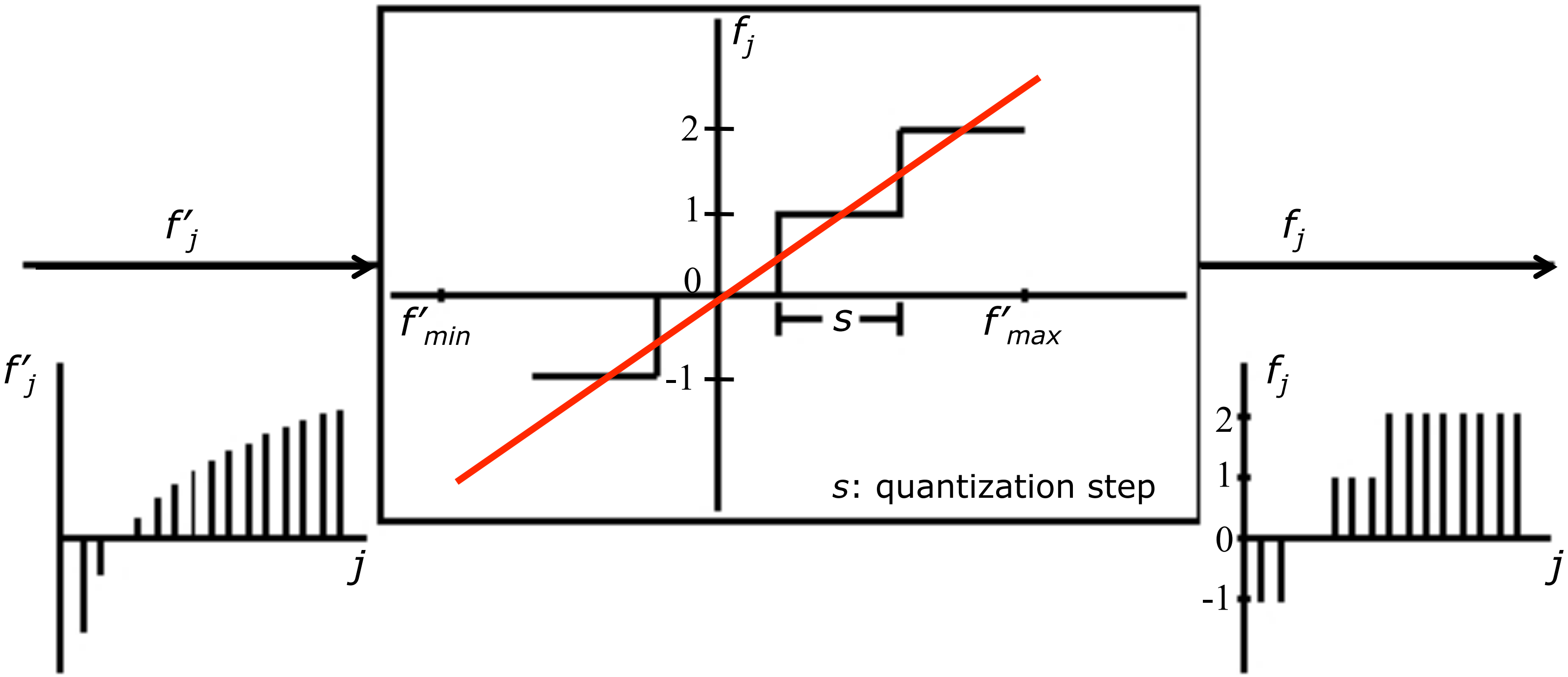




2. representing the amplitude values by a finite number of natural numbers:

   quantization

Quantization



- The number of quantization steps is defined by the number of bits we use to represent the value of the function.

Bits

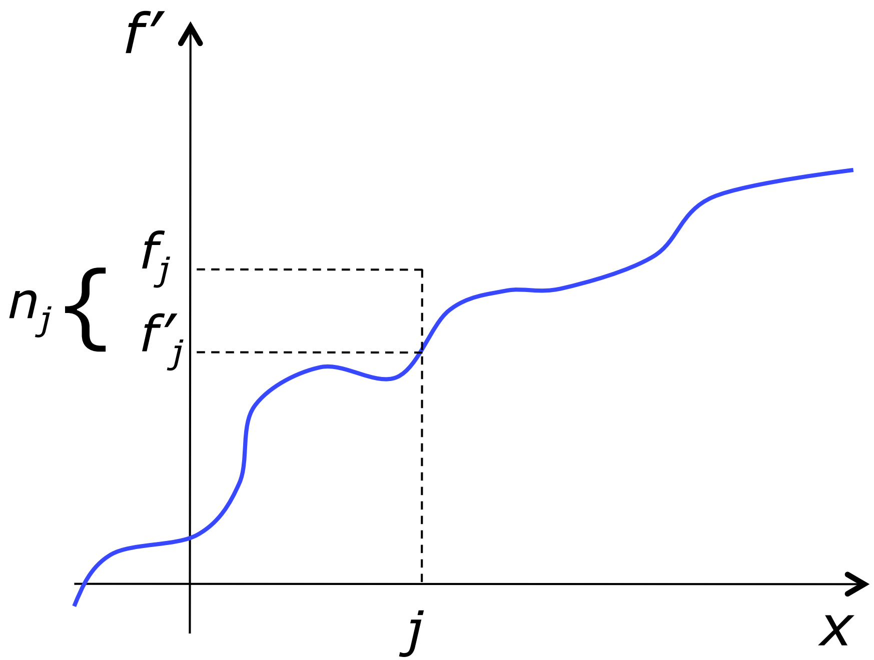
- Two key questions:
  1. How many bits?
  2. How do we use these bits?
- When we use B bits, we get 2^B quantized levels.
- Examples:
  - most intensity images: B = 8-12, 256 – 4,096 different gray values.
  - medical images: B = 10 - 16, 1024 – 65,536 different gray values.
  - most color images: B = 24-36, 8-12 for each color channel, at least 16 million colors.
- Typical data sizes for a 1024 x 1024 (1 MP) image:
  - at 8 bits => 1MB/img => a movie at 30fps creates 30MB/sec
  - at 12 bits => almost 1.6 MB/img => at 30 fps we get 47MB/sec
  - at 24 bits => 3.1 MB/img => at 30 fps we get 93MB/sec

    => a 5 minute movie needs 27GB.

Audio vs. Video Data Rates

Type	Specifications	Data Rate
Audio, understandable	1 channel, 8kHz @ 8 bits	64 kbit/sec
Audio, MPEG encoded	CD equivalence	384 kbit/sec
Audio, CD quality	2 channels, 44.1kHz @16 bits	1.4 Mbit/sec
Video, MPEG-2	640 × 480, 24 bits/pixel	0.42 MB/sec
Video, NTSC	640 × 480, 24 bits/pixel	27 MB/sec
Video, HDTV	1280 × 720, 24 bits/pixel	81 MB/sec

Quantization Error



- Quantization Error: The error we make when we approximate a real value f'_j by a discrete value f_j:

$$n_j = f'_j - f_j$$

Signal-to-Noise Ratio (SNR)

- There exists a standardized way of expressing the noise in a system or sensor that is associated with quantization. It is called the *Signal-to-Noise Ratio*.
- SNR is a general measure that is used for different types (sources) of noise.
- In Engineering SNR is a power ratio: $$SNR = \frac{P_{signal}}{P_{noise}}$$
- Within the context of pattern recognition, because of the uncertainty involved in the input signal, SNR is the ratio of the expected signal over the expected quantization noise.

$$SNR = \frac{E\{f'^2\}}{E\{n^2\}}$$

Signal-to-Noise Ratio (SNR) - continued

- The Signal-to-Noise Ratio is defined as:

$$SNR = r' = \frac{E\{f'^2\}}{E\{n^2\}}$$

where the quantization noise n is $n_j = f'_j - f_j$.

- The expected value $E\{\}$ is defined as:

$$E\{x\} = \int_{-\infty}^{\infty} x p(x) dx$$

where x is a random variable, and $p(x)$ is the probability density function (pdf) of x, which tells us how often different values of x occur.

- So, similar information on f' can guide us on how many bits to use.

SNR and logarithmic scale

- Because input signals can have a wide dynamic range, SNR is usually expressed in terms of the logarithmic decibel scale:

$$SNR_{dB} = r = 10\log_{10}\frac{E\{f'^2\}}{E\{n^2\}} = 10\log_{10}(r')$$

- Do we want a small or a large SNR? Why?

  Large is better.

  We want over 30dB SNR. Systems with 60dB are considered very good.

Does One Bit Make a Difference?

- Important question: How many bits should one use when quantizing a particular family of functions/ signals (i.e. medical images, or remote sensing data etc.)?

- Does one additional bit make a difference?

- Under certain assumptions (see next slide), the SNR is directly proportional to the number of bits used for quantization:

$$SNR_{db} = r = 6B - 7.2$$

- This means that 1 extra bit can increase the SNR by 6dB.

Assumptions

1. On average we have white noise.

$$E\{\vec{n}\} = 0 \text{ and } E\{\vec{n}\vec{n}^T\} = \sigma I$$

2. We have a signal with $E\{f'\} = 0$.
3. The error (noise) is uniformly distributed.
4. The signal values lie in a limited range:

$$-4\sigma_{f'} \leq f' < 4\sigma_{f'}$$

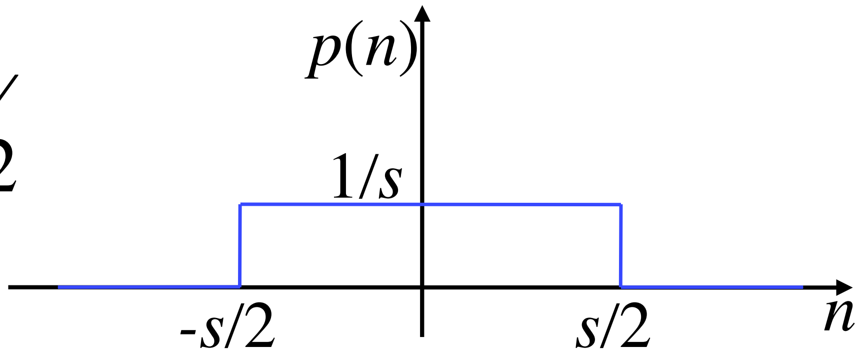
If we have a normal distribution, then
  about 68% of the values lie within 1 σ of the mean,
  about 95% of the values lie within 2 σ of the mean,
  about 99.7% of the values lie within 3 σ of the mean,
  about 99.99% of the values lie within 4 σ of the mean.

So **if** the values of f' follow a normal distribution, assumption 4 is reasonable.

Assumptions 1 and 3

- We have uniformly distributed white noise, $E\{n\} = 0$. Let s be the quantization step (quantization interval). Then $p(n)$ will be of the form:

$$p(n) = \begin{cases} \frac{1}{s} & \text{for } -\left(\frac{s}{2}\right) \le n \le \frac{s}{2} \\ 0 & \text{otherwise} \end{cases}$$



The width of the pdf has to be s and centered around the value 0 (since $E\{n\} = 0$), and the integral of the pdf has to sum up to 1 by definition.

SNR Denominator

- Recall that $SNR = r' = \dfrac{E\{f'^2\}}{E\{n^2\}}$
- What is $E\{n^2\}$?
- The definition of expected value is $E\{x\} = \int_{-\infty}^{\infty} x p(x) dx$.
- Thus, $E\{n^2\} = \int_{-\infty}^{\infty} n^2 p(n) dn$

$$= \int_{-s/2}^{s/2} n^2 p(n) dn = \frac{1}{s} \int_{-s/2}^{s/2} n^2 dn$$

$$= \frac{1}{s}\frac{1}{3}\left[n^3\right]_{-s/2}^{s/2} = \frac{1}{s}\frac{1}{3}\left(\frac{s^3}{8} - \left(-\frac{s^3}{8}\right)\right) = \frac{s^2}{12}$$

$$\boxed{E\{n^2\} = s^2/12} \qquad (1)$$

Assumption 2

- We have a signal with $E\{f'\}=0$.
- According to the definition of standard deviation:

$$\sigma_{f'} = \sqrt{E\{f'^2\} - \left(E\{f'\}\right)^2}$$

- However, by assumption 2, we get

$$\sigma_{f'} = \sqrt{E\{f'^2\}}$$

$$\boxed{\sigma_{f'}^2 = E\{f'^2\}} \quad (2)$$

Assumption 4

- The signal values lie in the range: $-4\sigma_{f'} \leq f' < 4\sigma_{f'}$
- So the length of the interval of the f' values is $8\sigma_{f'}$
- When we use B bits to store these $8\sigma_{f'}$ values, we have 2^B quantization levels.
- Assuming equidistant quantization, each quantization step, s, is

$$s = \frac{8\sigma_{f'}}{2^B} \qquad (3)$$

Assumption Combination

- So far, by exploiting the 4 assumptions we have shown:

$$E\{n^2\} = s^2/12 \qquad (1)$$

$$\sigma_{f'}^2 = E\{f'^2\} \qquad (2)$$

$$s = \frac{8\sigma_{f'}}{2^B} \qquad (3)$$

- From (1) and (3):

$$E\{n^2\} = \frac{2^6\sigma_{f'}^2}{12 \cdot 2^{2B}} \qquad (4)$$

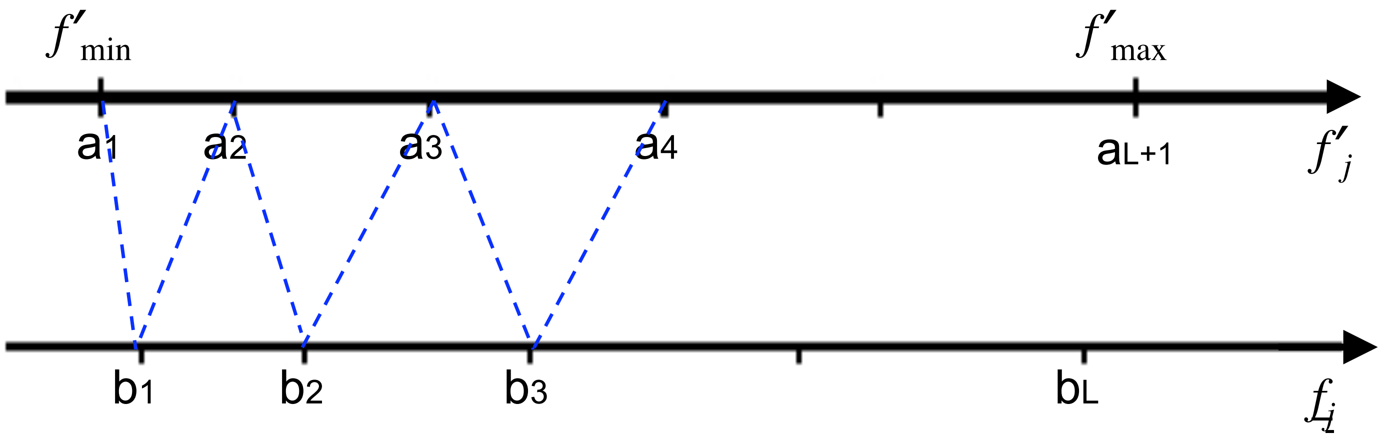
- Recall that SNR is defined as $r' = \frac{E\{f'^2\}}{E\{n^2\}}$

$$r' = \sigma_{f'}^2 \Big/ \left(\frac{2^6\sigma_{f'}^2}{12 \cdot 2^{2B}}\right) = \frac{12 \cdot 2^{2B}}{2^6} = 12 \cdot 2^{2B-6}$$

$$r = 10\log_{10} r' = 6B - 7.27$$

Mapping

- Using SNR as a criterion, we know how many bits to use, but how do we use them?
- To which discrete value do we map a continuous interval?



Good Mapping

- How can I tell whether my mapping is good?
- What is a possible objective function, a criterion to judge the quality of the mapping?
- Error measure (error that occurs when mapping f' to b_v)

$$\varepsilon = \sum_{v=1}^{L} \int_{a_v}^{a_{v+1}} (f' - b_v)^2 \, p(f') \, df'$$

- By weighing the error by the probability density of f', values that have a higher probability of occurring have a higher impact on the error term.
- The optimal quantization characteristics are defined by the values a_v , b_v which minimize the error ε.

Optimal Quantization Characteristics

- Optimal discrete value:

$$\partial \varepsilon / \partial b_v = \sum_{v=1}^{L} \int_{a_v}^{a_{v+1}} 2(f' - b_v) p(f') df' = 0 \qquad v = 1,2,\ldots,L$$

$$\int_{a_v}^{a_{v+1}} f' p(f') df' = b_v \int_{a_v}^{a_{v+1}} p(f') df' \Leftrightarrow b_v = \frac{\int_{a_v}^{a_{v+1}} f' p(f') df'}{\int_{a_v}^{a_{v+1}} p(f') df'}$$

- Optimal threshold level:

$$\partial \varepsilon / \partial a_v = \partial \left[\sum_{v=1}^{L} \int_{a_v}^{a_{v+1}} (f' - b_v)^2 p(f') df' \right] \Big/ \partial a_v = 0$$

Use the middle value as a threshold

$$(a_v - b_{v-1})^2 p(a_v) - (a_v - b_v)^2 p(a_v) = 0 \Leftrightarrow a_v = \frac{b_v + b_{v-1}}{2}$$

Pulse Code Modulation

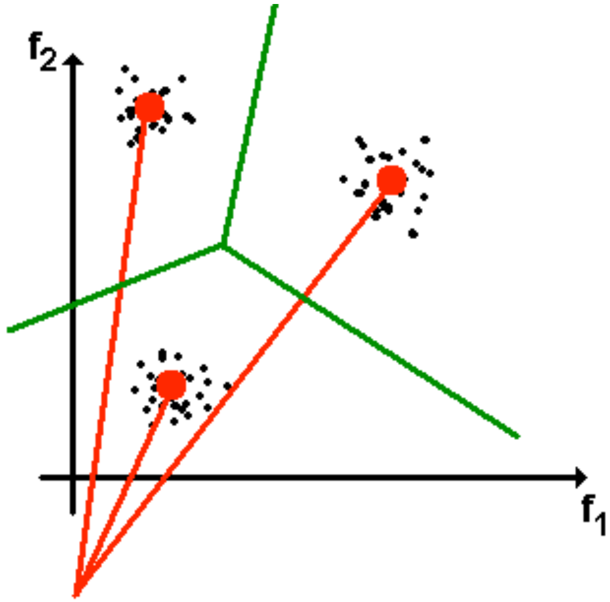
- A linear quantization characteristic function (with equally spaced quantization levels) is an optimal quantization if and only if the signal amplitudes are equally distributed.
- Coding using the methods introduced so far is called Pulse Code Modulation.
- Other coding methods, depending on the application are:
  - Coding with a minimal number of bits
  - Error detection and correction
  - Run-length encoding
  - Chain code

Vector Quantization

- So far, we have considered the quantization of real valued functions, i.e. $f \in R$.
- There exist signals where we have to deal with vector valued functions, $\vec{f} \in R^N$ (e.g. color images with RGB values).
- The quantization of vectors to discrete vectors is called *vector quantization*.
- **Vector quantization** is the process of mapping N-dimensional vectors in the vector space R^N into a finite set of vectors $Y = \{\vec{y}_i | i = 1 \cdots k\}$, where $k < N$.
- Each vector $\vec{y}_i$ is called a **code vector** or a **codeword**.
- The set of all the codewords, Y, is called a **codebook**.

Codebook Design

- There exist many vector quantization methods.
- We are just going to present one method which is based on mean values.
- Another one is based on computing nearest neighbor regions, aka Voronoi regions.



Using the Mean Vectors

- For each cluster in the training data compute the mean vector $\vec{\mu}_i$.
- Each mean vector $\vec{\mu}_i$ becomes the code vector or codeword, $\vec{y}_i$.
- All the mean vectors define the so-called code book, Y.
- Given an arbitrary input vector $\vec{f}'_j$ find the nearest code vector $\vec{y}_u$, s.t. $u = \min_i d(\vec{f}'_j, \vec{y}_i)$.
- Store the offset to the closest mean $\vec{y}_u$. There is a finite number of bits that can be used for the offset.
- Use your favorite distance metric, e.g. Euclidean, Manhattan, etc. We often use the Euclidean distance.

Computing the Codebook

- k-means algorithm
- k:# of code vectors
- Input: M data vectors $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_M, \vec{f}_i \in R^N$

1. Randomly assign the vectors $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_M$ to k clusters.
2. Compute the mean vector $\vec{\mu}_i$ for each cluster.
3. Reassign each vector $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_M$ to the cluster with the nearest mean vector $\vec{\mu}_i$.
4. Repeat 2. and 3. until no further changes occur

- Output: code book

Linde-Buzo-Gray Algorithm

- The Linde-Buzo-Gray (LBG) algorithm is a widely-used vector quantization algorithm which is very similar to the k-means algorithm.
- Main idea. Start with a single code vector. At each iteration, each code vector is split into two new vectors.

1. Initial state: compute the mean of the training data.
2. Initial estimation #1: code book of size 2.
3. Final estimation for code book of size 2, after training data reassignment.
4. Initial estimation #2: code book of size 4.
5. Final estimation for code book of size 4, after training data reassignment. …