Continuos wavelet transform (CWT) and edge detection

WTBV WS 2014/15

January 23, 2015

- Initial event:
  A. GROSSMANNund J. MORLET,
  Decompositions of Hardy functions into square integrable wavelets of constant shape, *SIAM J. Math. Analysis*, 1984
  (Analysis of seismic signals)
- ... but there were precursors . . . e.g.
  A. P. CALDÉRON,
  Intermediate Spaces and Interpolation, the Complex Method, *Studia Mathematica*, 1964
- see:
  S. JAFFARD, Y. MEYER, R. RYAN,
  Wavelets, Tools for Science and Technology, SIAM 2001,
  in particular: Chap. 2: Wavelets from a Historical Perspective

- Let $\psi : \mathbb{R} \to \mathbb{C}$ be a "suitable" wavelet function
- now: continuous dilation and translation of ψ

$$\psi_{s,a}(t) = \frac{1}{\sqrt{|s|}}\,\psi(\frac{t-a}{s}) \qquad (s, a \in \mathbb{R})$$

- *continuous wavlet transform* (CWT) of a signal $f : \mathbb{R} \to \mathbb{C}$ using $\psi(t)$ defined as

$$f^{\psi}(s,a) = \langle f, \psi_{s,a}\rangle = \int_{\mathbb{R}} f(t)\,\overline{\psi_{s,a}(t)}\,dt = \sqrt{|s|}\int_{\mathbb{R}} f(st+a)\,\overline{\psi(t)}\,dt$$

- Intuitively: $f^{\psi}(s,a)$ represents the behavior of $f(t)$ in the vicinity of $a \in \mathbb{R}$ in resolution (scaling) $s \in \mathbb{R}$:

$$\|f(t) - \psi_{s,a}(t)\|^2 = \|f(t)\|^2 + \|\psi_{s,a}(t)\|^2 - 2\,\Re\left[f^{\psi}(s,a)\right]$$

Only the $\Re$-term depends on s and a !
Minimizing $\|f(t) - \psi_{s,a}(t)\|^2$ means maximizing $\Re \ldots$

- Let $\psi(t)$ be a wavelet function with $\|\psi\|^2 = 1$ (w.l.o.g.), then $t \mapsto |\psi(t)|^2$ can be viewed as a probability density on $\mathbb{R}$ with average μ and variance σ^2:

$$\mu = \int t\,|\psi(t)|^2\,dt \qquad \sigma^2 = \int (t-\mu)^2\,|\psi(t)|^2\,dt$$

- Parseval-Plancherel: $\|\widehat{\psi}\|^2 = \|\psi(t)\|^2 = 1$

  Also $\lambda \mapsto |\widehat{\psi}(\lambda)|^2$ is a probability density with average $\widehat{\mu}$ and variance $\widehat{\sigma}^2$

$$\widehat{\mu} = \int \lambda\,|\widehat{\psi}(\lambda)|^2\,d\lambda \qquad \widehat{\sigma}^2 = \int (\lambda-\widehat{\mu})^2\,|\widehat{\psi}(\lambda)|^2\,d\lambda$$

- For $s > 0$, $a \in \mathbb{R}$ one has

$$\|\widehat{\psi_{s,a}}(t)\|^2 = \|\psi_{s,a}(t)\|^2 = \|\psi(t)\|^2 = 1$$
$$\widehat{\psi_{s,a}}(\lambda) = \sqrt{s}e^{-2\pi i a\lambda}\widehat{\psi}(s\lambda)$$

- Localization in the time domain

$$\mu_{s,a} = \int t\,|\psi_{s,a}(t)|^2\,dt = \ldots = s\,\mu + a$$
$$\sigma_{s,a}^2 = \int (t - \mu_{s,a})^2\,|\psi_{s,a}|^2\,dt = \ldots = s^2\,\sigma^2$$

- Localization in the frequency domain

$$\widehat{\mu}_{s,a} = \int t\,|\widehat{\psi_{s,a}}(t)|^2\,dt = \ldots = \frac{1}{s}\,\widehat{\mu}$$
$$\widehat{\sigma}_{s,a}^2 = \int (t - \mu_{s,a})^2\,|\widehat{\psi_{s,a}}|^2\,dt = \ldots = \frac{1}{s^2}\,\widehat{\sigma}^2$$

- The "uncertainty" $\sigma_{s,a}^2 \cdot \widehat{\sigma}_{s,a}^2$ is independent of s and a

$$\sigma_{s,a}^2 \cdot \widehat{\sigma}_{s,a}^2 = \sigma^2 \cdot \widehat{\sigma}^2$$

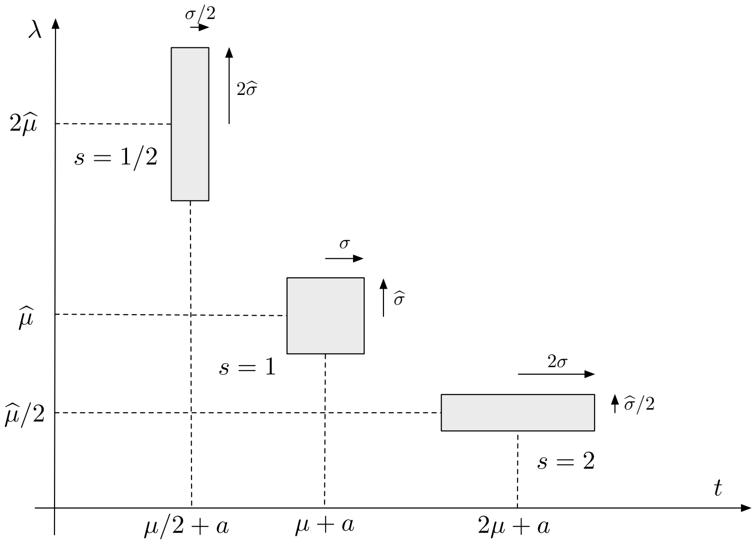

Figure: Heisenberg boxes for $\psi_{s,a}$, $s = 1/2, 1, 2$

- HAAR wavelet function

$$\psi_{haar}(t) = \begin{cases} 1 & 0 \leq t < 1/2 \\ -1 & 1/2 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- *mexican-hat* wavelet

$$\psi_{mex}(t) = (1 - 2\,t^2)\, e^{-t^2}$$

- MORLET wavelet

$$\psi_{mor}(t) = e^{-t^2} \cos\left(\pi \sqrt{\frac{2}{\ln 2}}\, t\right)$$

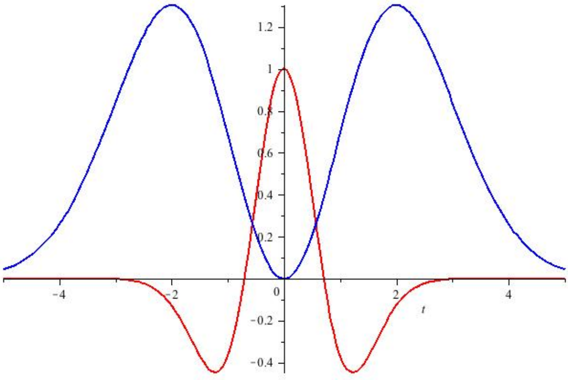

Figure: mexican-hat wavelet (in red) and its spectrum (in blue)

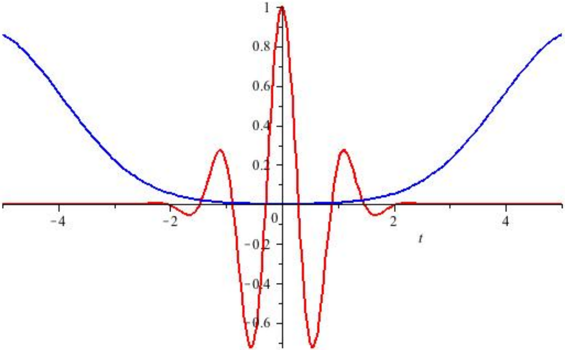

Figure: MORLET wavelet (in red and its spectrum (in blue)

- Fourier transforms

$$\widehat{f_{haar}}(s) = \frac{4\,i\,(\sin(1/4\,s))^2\,\mathrm{e}^{-1/2\,is}}{s} \qquad \widehat{f_{haar}}(0) = 0$$

$$\widehat{f_{mex}}(s) = 1/2\,s^2 \mathrm{e}^{-1/4\,s^2}\sqrt{\pi} \qquad \widehat{f_{mex}}(0) = 0$$

$$\widehat{f_{mor}}(s) = \sqrt{\pi}\cosh\left(1/2\,\frac{s\pi\sqrt{2}}{\sqrt{\ln(2)}}\right)\mathrm{e}^{-1/4\,s^2 - 1/2\,\frac{\pi^2}{\ln(2)}} \qquad \widehat{f_{mor}}(0) \approx 0.0014$$

- *admissibility* constants

$$C_{haar} = \int_{s=-\infty}^{\infty} \frac{|\widehat{f_{haar}}(s)|^2}{|s|}\,ds = 2\,\ln(2)$$

$$C_{mex} = \int_{s=-\infty}^{\infty} \frac{|\widehat{f_{mex}}(s)|^2}{|s|}\,ds = \pi$$

$$C_{mor} = \int_{s=-\infty}^{\infty} \frac{|\widehat{f_{mor}}(s)|^2}{|s|}\,ds = \infty$$

- Intuitively:

$$f^{\psi}(s,a) = \int_{t=-\infty}^{\infty} f(t)\, \frac{1}{\sqrt{|s|}}\, \psi(\frac{t-a}{s})\, dt$$

  represents the behavior of $f(t)$ in the vicinity of $a \in \mathbb{R}$ in resolution (scaling) $s \in \mathbb{R}$
- The data

$$\Big(f^{\psi}(s,a) \Big)_{s>0, a\in\mathbb{R}}$$

  give a highly redundant representation of the function $f(t)$
- Problem: how can one recover $f(t)$ from these data ?

- CALDÉRON's reconstruction formula:

$$f(t) = \frac{1}{C_\psi} \int_{s\in\mathbb{R}} \int_{a\in\mathbb{R}} f^\psi(s,a)\, \psi_{s,a}(t)\, da\, \frac{ds}{s^2}$$

where

$$0 < C_\psi = \int_{\lambda\in\mathbb{R}} \frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|}\, d\lambda < \infty$$

- Note that the condition $C_\psi < \infty$ implies

$$\int_{\mathbb{R}} \psi(t)\, dt = \widehat{\psi}(0) = 0$$

- If $\psi : \mathbb{R} \to \mathbb{R}$ is a real wavelet function, then CALDÉRON's formula can be written as

$$f(t) = \frac{1}{C'_\psi} \int_{s>0} \int_{a\in\mathbb{R}} f^\psi(s,a)\,\psi_{s,a}(t)\,da\,\frac{ds}{s^2}$$

where

$$0 < C'_\psi = \int_{\lambda>0} \frac{|\widehat{\psi}(\lambda)|^2}{\lambda}\,d\lambda < \infty$$

- This simplification is justified by the symmetry property

$$\overline{\widehat{\psi}(\lambda)} = \widehat{\psi}(-\lambda)$$

for any real function $\psi(t)$

- Lemma (1) [Fourier transform w.r.t. t]

$$\left[\,\psi_{s,a}(t)\,\right]^{\wedge_t}(\lambda) = \sqrt{|s|}\,e^{-2\pi i a\lambda}\,\widehat{\psi}(\lambda s)$$

- Lemma (2) [Fourier transform w.r.t. a]

$$\left[\,\overline{\psi_{s,a}}(t)\,\right]^{\wedge_a}(\lambda) = \frac{s}{\sqrt{|s|}}\,e^{-2\pi i t\lambda}\,\overline{\widehat{\psi}(\lambda s)}$$

- Lemma (3) [Fourier transform w.r.t. a]

$$\left[\,f^{\psi}(s,a)\,\right]^{\wedge_a}(\lambda) = \frac{s}{\sqrt{|s|}}\,\widehat{f}(\lambda)\,\overline{\widehat{\psi}(\lambda s)}$$

- Consequence of Lemma (3):

$$f^{\psi}(s,a) = \left[\frac{s}{\sqrt{|s|}} \, \widehat{f}(\lambda) \, \overline{\widehat{\psi}(\lambda s)} \right]^{\vee_a}$$
$$= \frac{s}{\sqrt{|s|}} \int_{\mathbb{R}} \widehat{f}(\lambda) \, \overline{\widehat{\psi}(\lambda s)} \, e^{2\pi i a \lambda} d\lambda$$

- This indicates an efficient way for computing the wavelet coefficients $f^{\psi}(s,a)$ based on the FFT:
  1. compute $\widehat{f}(\lambda)$
  2. compute $\widehat{\psi}(\lambda)$
     (NB $\widehat{\psi}$ is explicitly known in many cases)
  3. multiply $\widehat{f}(\lambda) \cdot \overline{\widehat{\psi}(\lambda s)}$
  4. apply the inverse FFT

Proof (sketch) of CALDÉRON's reconstruction formula

- From Parseval-Plancherel and Lemmas (2) and (3) one gets

$$\begin{aligned}\int_{a\in\mathbb{R}} f^{\psi}(s,a)\,\psi_{s,a}(t)\,da &= \langle f^{\psi}(s,a), \overline{\psi_{s,a}}(t)\rangle_a = \\ &= \langle\, [f^{\psi}(s,a)]^{\wedge_a}(\lambda), [\overline{\psi_{s,a}}(t)]^{\wedge_a}(\lambda)\,\rangle_\lambda \\ &= \frac{s^2}{|s|}\cdot\langle \widehat{f}(\lambda)\,\overline{\widehat{\psi}(\lambda s)}, e^{-2\pi i t\lambda}\,\overline{\widehat{\psi}(\lambda s)}\rangle_\lambda \\ &= \frac{s^2}{|s|}\cdot\int_{\lambda\in\mathbb{R}} \widehat{f}(\lambda)\,|\widehat{\psi}(\lambda s)|^2\,e^{2\pi i t\lambda}\,d\lambda\end{aligned}$$

- and then

$$\begin{aligned}\int_{s\in\mathbb{R}}\int_{a\in\mathbb{R}} f^{\psi}(s,a)\,\psi_{s,a}(t)\,da\,\frac{ds}{s^2} &= \int_{\lambda\in\mathbb{R}} \widehat{f}(\lambda)\,e^{2\pi i t\lambda}\int_s \frac{|\widehat{\psi}(\lambda s)|^2}{|s|}\,ds\,d\lambda \\ &= \int_{\lambda} \widehat{f}(\lambda)\,e^{2\pi i t\lambda}\int_s \frac{|\widehat{\psi}(s)|^2}{|s|}\,ds\,d\lambda \\ &= C_\psi\cdot\int_{\lambda\in\mathbb{R}} \widehat{f}(\lambda)\,e^{2\pi i t\lambda} d\lambda \\ &= C_\psi\cdot f(t)\end{aligned}$$

- Theorem
  1. If $\psi(t)$ is a continuous function with
$$\int_{t\in\mathbb{R}} \psi(t)\,dt = 0$$
  2. and if there are positive constants A, B s.th.
$$|\psi(t)| \leq A\,e^{-B\,|t|} \quad (t \in \mathbb{R})$$
     (exponentially rapid vanishing at infinity)

  then
$$C_\psi = \int_{\lambda\in\mathbb{R}} \frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|}\,d\lambda < \infty$$
  and CALDÉRON's reconstruction formula holds for all $f \in \mathcal{L}^2(\mathbb{R})$

- Remarks on condition $C_\psi < \infty$
  - Eponentially rapid vanishing of $\psi(t)$ at infinity implies $\psi(t) \in \mathcal{L}^2(\mathbb{R})$ and $\widehat{\psi}(\lambda) \in \mathcal{L}^2(\mathbb{R})$ and $\widehat{\psi}(\lambda) \in \mathcal{C}^1(\mathbb{R})$ (differentiability)
  - Decompose the integral into two parts

$$C_\psi = \int_{\lambda \in \mathbb{R}} \frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|}\, d\lambda = \int_{|\lambda| \le 1} \ldots + \int_{|\lambda| \ge 1} \ldots$$

  - Taylor expansion of $\widehat{\psi}(\lambda)$ at $\lambda = 0$ and

$$\widehat{\psi}(0) = \int \psi(t)\, dt = 0$$

    shows that the first integral $\int_{|\lambda| \le 1} \ldots$ is finite
  - As for the second integral,

$$\int_{|\lambda| \ge 1} \ldots \le \int |\widehat{\psi}(\lambda)|^2\, d\lambda \le \|\widehat{\psi}\|^2 < \infty$$

    shows that this is finite too

- The HAAR wavelet function $\psi_{haar}(t)$ can be regarded as a derivative

$$\psi_{haar}(t) = \frac{d}{dt}\Delta(t) \quad \text{mit} \quad \Delta(t) = \begin{cases} t & 0 \le t \le 1/2 \\ 1 - t & 1/2 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- The mexican-hat wavelet function $\psi_{mex}(t)$ is a derivative

$$\psi_{mex}(t) = \frac{d}{dt}\left(t\, e^{-t^2}\right) = \frac{d^2}{dt^2}\frac{-e^{-t^2}}{2}$$

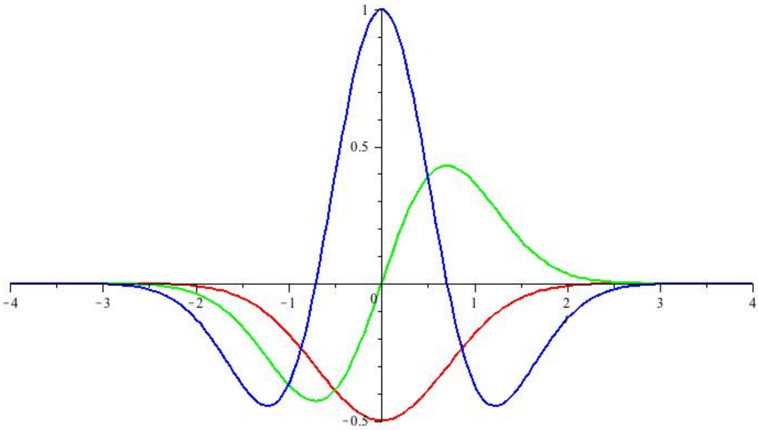

Figure: mexican-hat wavelet as second derivative of a Gaussian

- Let $\psi(t)$ be a wavelet function in the sense of the Theorem
- Let $\psi(t)$ be the derivative of a "smoothing function" $\theta(t)$

$$\psi(t) = \frac{d}{dt}\theta(t)$$

- Scaling of $\theta(t)$

$$\overleftarrow{\theta_s}(t) = \frac{1}{s}\theta(-\frac{t}{s})$$

- Then

$$(*) \qquad f^{\psi}(s,a) = -s^{-3/2}\frac{d}{da}(f \star \overleftarrow{\theta_s})(a)$$

- Note: $f \star \overleftarrow{\theta_s}$ is a $\overleftarrow{\theta_s}$-smoothed version version of f

- Interpretation:
  Edges in the graph of $f(t)$ can be recognized by absolutely large values of the wavelet coefficients $f^{\psi}(s,a)$ over many scales (s values)
- Proof of $(*)$

  We have

$$(f \star \overleftarrow{\theta_s})(a) = \int_{t\in\mathbb{R}} f(t)\,\overleftarrow{\theta_s}(a-t)\,dt = \int_{t\in\mathbb{R}} f(t)\,\frac{1}{s}\,\theta(\frac{t-a}{s})\,dt$$

  and hence

$$\frac{d}{da}(f \star \overleftarrow{\theta_s})(a) = \int_{t\in\mathbb{R}} f(t)\,\frac{1}{s}\,\theta'(\frac{t-a}{s})\,(-\frac{1}{s})\,dt$$
$$= \int_{t\in\mathbb{R}} f(t)\,(-\frac{1}{s^2})\,\psi(\frac{t-a}{s})\;dt = -s^{-3/2}\langle f, \psi_{s,a}\rangle$$

- Assume that the wavelet function $\psi(t)$ belongs to a MRA with scaling function $\phi(t)$
- Scaling and wavelet identities are

$$\phi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \, \phi(2t - k)$$

$$\psi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} g_k \, \phi(2t - k)$$

- Approximation and detail coefficients of a function $f(t)$, using dyadic scaling and integer translation $(s, a) = (2^m, n)$, are

$$a_{m,n} = \langle f, \phi_{2^m,n} \rangle \qquad d_{m,n} = \langle f, \psi_{2^m,n} \rangle$$

- Recursion formulas

$$\phi_{2^{m+1},n}(t) = 2^{-(m+1)/2} \phi\left(\frac{t-n}{2^{m+1}}\right) = \ldots = \sum_k h_k \, \phi_{2^m, n+k\,2^m}(t)$$

$$\psi_{2^{m+1},n}(t) = 2^{-(m+1)/2} \psi\left(\frac{t-n}{2^{m+1}}\right) = \ldots = \sum_k g_k \, \phi_{2^m, n+k\,2^m}(t)$$

- Recursion formulas for approximation and wavelet coefficients

$$a_{m+1,n} = \sum_{k\in\mathbb{Z}} h_k \, a_{m,n+k\,2^m} \quad (n \in \mathbb{Z})$$

$$d_{m+1,n} = \sum_{k\in\mathbb{Z}} g_k \, a_{m,n+k\,2^m} \quad (n \in \mathbb{Z})$$

- Written as filtering operations

$$\left(a_{m+1,n}\right)_{n\in\mathbb{Z}} = \overleftarrow{\left[(\uparrow_2)^m \boldsymbol{h}\right]} \star \left(a_{m,n}\right)_{n\in\mathbb{Z}}$$

$$\left(d_{m+1,n}\right)_{n\in\mathbb{Z}} = \overleftarrow{\left[(\uparrow_2)^m \boldsymbol{g}\right]} \star \left(a_{m,n}\right)_{n\in\mathbb{Z}}$$

- Here $(\uparrow_2)^m \boldsymbol{h}$ is the filter constructed from $\boldsymbol{h}$ by using m-fold upsampling with factor 2
- Algorithmic realization *algorithme à trous*
- M. Holschneider et al., A real-time algorithm for signal analysis with the help of wavelet transform. In: *Wavelets, Time-Frequency Methods and Phase Space*, Springer-Verlag, 1989

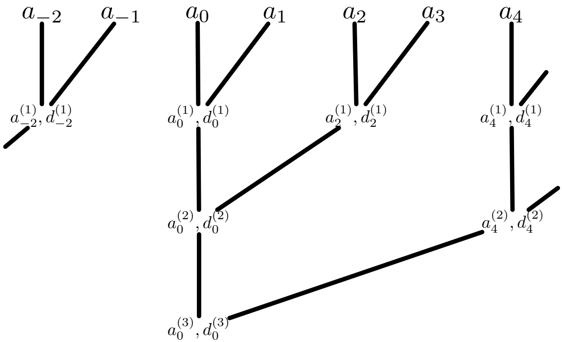

Figure: Scheme of the Haar transform

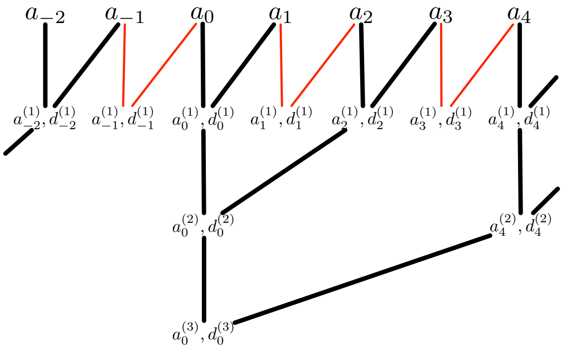

Figure: *à-trous* scheme (one level) for the Haar transform

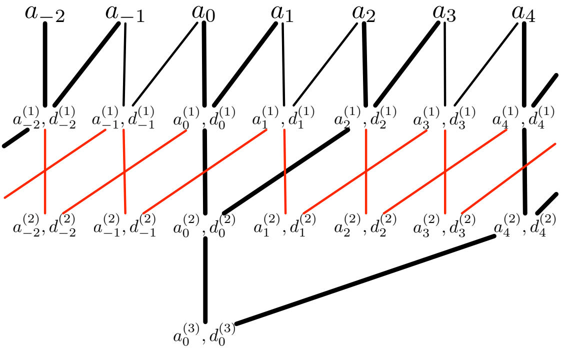

Figure: *à-trous* scheme (two levels) for the Haar transform

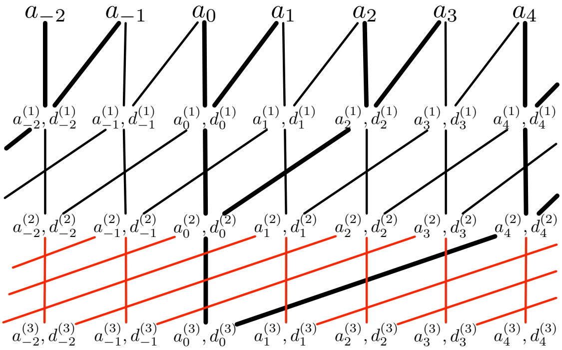

Figure: *à-trous* scheme (three levels) for the Haar transform

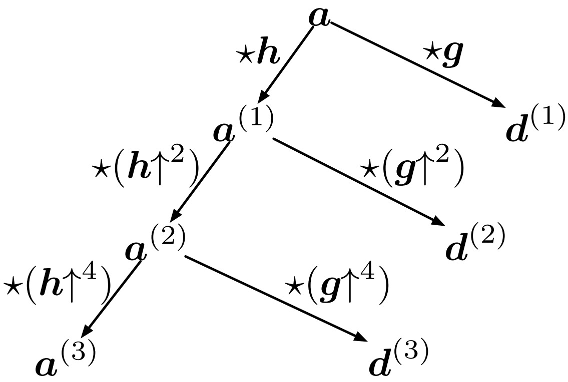

Figure: *à-trous* scheme (three levels)

high-pass filter: $\boldsymbol{g}$, low-pass filter: $\boldsymbol{h}$, signal: $\boldsymbol{a} = (a_k)_{n\in\mathbb{Z}}$
filtered signals: $\boldsymbol{a}^{(k)} = \left(a_n^{(k)}\right)_{n\in\mathbb{Z}}$, $\boldsymbol{d}^{(k)} = \left(d_n^{(k)}\right)_{n\in\mathbb{Z}}$,

- Let $\psi(x)$ be a one-dimension wavelet function
- $\Psi(x, y) = \psi(x)\,\psi(y)$ the tw0-dimensional separable wavelet function constructed from it
- The 2-dim. CWT of a function $f(x, y)$ is

$$f^{\Psi}(a, b, s) = \frac{1}{s}\iint_{x,y\in\mathbb{R}\times\mathbb{R}} f(x, y)\,\Psi(\frac{x-a}{s}, \frac{y-b}{s})\,dx\,dy$$

- Let $\psi(x) = \frac{d}{dx}\,\theta(x)$ be the derivative of a "smoothing function" $\theta(x)$
- 2-dim separable smoothing function

$$\Theta(x, y) = \theta(x)\,\theta(y)$$

- 2-dim partial wavelet functions

$$\Psi^x(x, y) = \psi(x)\,\theta(y) = \frac{\partial}{\partial x}\,\Theta(x, y)$$
$$\Psi^y(x, y) = \theta(x)\,\psi(y) = \frac{\partial}{\partial y}\,\Theta(x, y)$$

- 2-dim partial CWT

$$
\begin{aligned}
f^{\Psi^x}(a,b,s) &= \frac{1}{s}\iint_{x,y\in\mathbb{R}\times\mathbb{R}} f(x,y)\,\Psi^x(\frac{x-a}{s},\frac{y-b}{s})\,dx\,dy \\
&= -\frac{\partial}{\partial a}\iint_{x,y\in\mathbb{R}\times\mathbb{R}} f(x,y)\,\Theta(\frac{x-a}{s},\frac{y-b}{s})\,dx\,dy \\
f^{\Psi^y}(a,b,s) &= \frac{1}{s}\iint_{x,y\in\mathbb{R}\times\mathbb{R}} f(x,y)\,\Psi^y(\frac{x-a}{s},\frac{y-b}{s})\,dx\,dy \\
&= -\frac{\partial}{\partial b}\iint_{x,y\in\mathbb{R}\times\mathbb{R}} f(x,y)\,\Theta(\frac{x-a}{s},\frac{y-b}{s})\,dx\,dy
\end{aligned}
$$

- The integral $\iint \ldots$ is essentially scaled-Θ-smoothed version of f
- $(-f^{\Psi^x}(a,b,s), -f^{\Psi^y}(a,b,s))$ is the *gradient* (a,b) of this function

Recall CANNY's definition

- Let $f \in \mathcal{L}^2(\mathbb{R}^2)$.
  The vertex $(x_0, y_0) \in \mathbb{R}^2$ is an *edge vertex* of $f(x, y)$ if

$$|\mathrm{grad} f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

  has a local maximum when passing through (x_0, y_0) in the direction of $(\mathrm{grad} f)(x_0, y_0)$

- This can be tested by computing

$$\left(f^{\Psi^x}(a, b, s)\right)^2 + \left(f^{\Psi^y}(a, b, s)\right)^2$$

  over several scale values s

- A vertex which is declared edge vertex over several scales is assumed to be a true edge vertex

Looking at this in the MRA context

- Scaling, wavelet and smoothing (1-dim) are described by

$$\phi(x) = \sqrt{2}\sum_k h_k\,\phi(2x-k) \qquad\qquad \psi(x) = \sqrt{2}\sum_k g_k\,\phi(2x-k)$$

$$\theta(x) = \sqrt{2}\sum_\ell r_k\,\theta(2x-\ell)$$

- Scalierung and wavelet equations for $\Phi^x(x,y) = \phi(x)\,\theta(y/2)$ and for $\Psi^x(x,y) = \psi(x)\,\theta(y)$ are

$$\Phi^x(x,y) = 2\sum_{k,\ell} h_k\,r_\ell\,\Phi^x(2x-k,2y-\ell)$$

$$\Psi^x(x,y) = 2\sum_{k,\ell} g_k\,\epsilon_\ell\,\Phi^x(2x-k,2y-\ell)$$

where $\epsilon_\ell = \frac{1}{\sqrt{2}}\,\delta_{\ell,0}$.
Similarly for $\Phi^y(x,y)$ and $\Psi^y(x,y)$

- The HAAR wavelet function $\psi_{haar}(t)$ is the derivative of the smoothing function $\theta(t) = \Delta(t)$:

$$\psi_{haar}(t) = \frac{d}{dt}\Delta(t) \quad \text{where} \quad \Delta(t) = \begin{cases} t & 0 \leq t \leq 1/2 \\ 1 - t & 1/2 \leq t \leq 1 \\ 0 & \text{sonst} \end{cases}$$

- The function $\Delta(t)$ satisfies

$$\Delta(x) + 2\,\Delta(x - 1/2) + \Delta(x - 1) = 2\Delta(x/2)$$

- which can be written as a scaling equation

$$\Delta(x) = \frac{1}{2}\left(\Delta(2x) + 2\,\Delta(2x - 1) + \Delta(2x - 2)\right)$$

- so that

$$\boldsymbol{r} = \frac{1}{2\sqrt{2}}\,\langle 1, 2, 1 \rangle$$

- Approximation and detail coefficients are

$$a^x_{m;k,\ell} = \langle f, \Phi^x_{2^m,k,\ell} \rangle = \iint f(x,y) \frac{1}{2^m} \Phi^x\Big(\frac{x-k}{2^m}, \frac{y-\ell}{2^m}\Big)\, dx\, dy$$

$$d^x_{m;k,\ell} = \langle f, \Psi^x_{2^m,k,\ell} \rangle = \iint f(x,y) \frac{1}{2^m} \Psi^x\Big(\frac{x-k}{2^m}, \frac{y-\ell}{2^m}\Big)\, dx\, dy$$

  and analogously for $a^y_{m;k,\ell}$ and $d^y_{m;k,\ell}$
- Recursions formula for approximation

$$a^x_{m+1;p,q} = \sum_{k,\ell} h_k\, r_\ell\, a^x_{m;p+k2^m,q+\ell 2^m}$$

- detail coefficients

$$d^x_{m+1;p,q} = \sum_{k,\ell} g_k\, \epsilon_\ell\, a^x_{m;p+k2^m,q+\ell 2^m} = \frac{1}{\sqrt{2}} \sum_k g_k\, a^x_{m;p+k2^m,q}$$

- Formulas for $a^x_{m;k,\ell}$ and $d^y_{m;k,\ell}$ are analogous

- Scheme for computation (*à trous* algorithm)

$$A_m^x = \left[f^{\Phi^x}(2^m; p, q) \right]_{p,q} \qquad A_m^y = \left[f^{\Phi^y}(2^m; p, q) \right]_{p,q}$$

$$D_m^x = \left[f^{\Psi^x}(2^m; p, q) \right]_{p,q} \qquad D_m^y = \left[f^{\Psi^y}(2^m; p, q) \right]_{p,q}$$

where $A_0 = A_0^x = A_0^y = [f(p, q)]_{p,q}$