From filters to scaling and wavelet functions via the cascade algorithm or via dyadic interpolation

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- Motivation: Recall the Haar situation
  - Haar scaling and wavelet functions

$$\phi(t) = \chi_{[0,1)}(t) \quad \psi(t) = \chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t)$$

  - scaled and translated Haar functions

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k) \quad \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$$

  - Haar wavelet coefficients

$$a_{j,k} = \langle\, f(t) \,|\, \phi_{j,k}(t) \,\rangle \quad d_{j,k} = \langle\, f(t) \,|\, \psi_{j,k}(t) \,\rangle$$

  - Haar scaling and wavelet equations

$$\phi_{j,k}(t) = \frac{1}{\sqrt{2}} \left(\phi_{j+1,2k}(t) + \phi_{j+1,2k+1}(t)\right)$$
$$\psi_{j,k}(t) = \frac{1}{\sqrt{2}} \left(\phi_{j+1,2k}(t) - \phi_{j+1,2k+1}(t)\right)$$

  - Haar coefficient equations

$$a_{j,k}(t) = \frac{1}{\sqrt{2}} \left(a_{j+1,2k}(t) + a_{j+1,2k+1}(t)\right)$$
$$d_{j,k}(t) = \frac{1}{\sqrt{2}} \left(a_{j+1,2k}(t) - a_{j+1,2k+1}(t)\right)$$

- Motivation (contd.):
  - Haar filters

$$\boldsymbol{h} = (h_0, h_1) = \frac{1}{\sqrt{2}}(1,1) \quad \boldsymbol{g} = (g_0, g_1) = \frac{1}{\sqrt{2}}(1,-1)$$

  - Haar scaling and wavelet equations again

$$\begin{aligned}\phi(t) = \phi_{0,0}(t) &= \frac{1}{\sqrt{2}}\left(\phi_{1,0}(t) + \phi_{1,1}(t)\right) \\ &= \sum_{k=0}^{1} h_k\,\phi_{1,k}(t) = \sqrt{2}\sum_{k=0}^{1} h_k\,\phi(2t-k)\end{aligned}$$

$$\begin{aligned}\psi(t) = \psi_{0,0}(t) &= \frac{1}{\sqrt{2}}\left(\phi_{1,0}(t) - \phi_{1,1}(t)\right) \\ &= \sum_{k=0}^{1} g_k\,\phi_{1,k}(t) = \sqrt{2}\sum_{k=0}^{1} g_k\,\phi(2t-k)\end{aligned}$$

- Motivation (contd.)
  - Optimal approximation of functions by step functions

$$P_j : \mathcal{L}^2 \to \mathcal{V}_j : f(t) \to \sum_k a_{j,k}\, \phi_{j,k}(t) = 2^{j/2} \sum_k a_{j,k}\, \phi(2^j t - k)$$

  - The translates

$$\left\{ \phi_{j,k}(t) \right\}_{k \in \mathbb{Z}} = \left\{ 2^{j/2} \phi(2^j t - k) \right\}_{k \in \mathbb{Z}}$$

    are an orthogonal basis of the approximation space $\mathcal{V}_j$
  - Detail information for optimal approximation of functions by step functions

$$Q_j : \mathcal{L}^2 \to \mathcal{W}_j : f(t) \to \sum_k d_{j,k}\, \psi_{j,k}(t) = 2^{j/2} \sum_k d_{j,k}\, \psi(2^j t - k)$$

  - The translates

$$\left\{ \psi_{j,k}(t) \right\}_{k \in \mathbb{Z}} = \left\{ 2^{j/2} \psi(2^j t - k) \right\}_{k \in \mathbb{Z}}$$

    are an orthogonal basis of the detail space $\mathcal{W}_j$

- $\boldsymbol{h} = (h_0, h_1, \ldots, h_L)$ a finite filter satisfying the orthogonality conditions
- The *scaling identity*

$$(S) \qquad \phi(t) = \sum_{k=0}^{L} h_k \, \phi_{1,k}(t) = \sqrt{2} \sum_{k=0}^{L} h_k \, \phi(2t - k)$$

- Question: is there a “reasonable” *scaling function* $\phi(t) : \mathbb{R} \to \mathbb{R}$ satisfying this equation?
- Comment: The scaling identity (S) has “self-referential” character, One cannot expect that a function satisfying (S) can be described by a more or less simple analytical expression. One rather expects a “fractal” object. One may try to get an idea via an iterative construction approximating it

- Once exact or approximate values for $\phi(t)$ at sufficiently many positions are kown, one may use the *wavelet identity*

$$(W) \qquad \psi(t) = \sum_{k=0}^{L} g_k \, \phi_{1,k}(t) = \sqrt{2} \sum_{k=0}^{L} (-1)^{L-k} h_k \, \phi(2t + k - L)$$

with $g_k = (-1)^k h_{L-k}$ to get an approximate idea of the corresponding wavelet function $\psi(t)$

- Identity (S) can be considered as a *fixed-point equation* for the function $\phi(t)$ to be determined
- This leads to an iterative procedure for computing functions $\phi^{[n]}(t)$ $(n = 0, 1, 2, \ldots)$:
  - start with $\phi^{[0]}(t) = \chi_{[0,1)}(t)$
  - for $n = 0, 1, 2, \ldots$ compute $\phi^{[n+1]}(t)$ from $\phi^{[n]}(t)$ by setting

$$\phi^{[n+1]}(t) = \sqrt{2} \sum_{k=0}^{L} h_k \, \phi^{[n]}(2t - k)$$

- One expects that under appropriate conditions the sequence $\left(\phi^{[n]}(t)\right)_{n \geq 0}$ will converge in the $\mathcal{L}^2$-norm towards a function $\phi(t) \in \mathcal{L}^2(\mathbb{R})$, the *scaling function*, which then satisfies (S)
- This hope can indeed be justified rigorously under rather weak conditions

- Finiteness of $\boldsymbol{h}$ guarantees that all approximating functions $\phi^{[n]}(t)$ vanish outside the interval $[0, L]$
- Hence the same holds true for the the limit function $\phi(t)$
- The wavelet function can be approximated by defining

$$\psi^{[n]}(t) = \sqrt{2} \sum_{k=0}^{L} g_k \, \phi^{[n]}(t) \quad (n \geq 0)$$

- One expects that the functions $\psi^{[n]}(t)$ converge in the $\mathcal{L}^2$-norm towards the true *wavelet function* $\psi(t)$
- For more information on how to justify this limiting procedure see the Lecture Notes (Section 9)
- Section 12 of the Lecture Notes contains several examples illustrating this iterative procedure

- An alternative approach starts by computing the true exact values of $\phi(t)$ for positions $t \in \{0, 1, 2, \ldots, L\}$.
  This is done bei solving an eigenvalue problem obtained from the scaling identity (S)
- Then for $j = 1, 2, 3, \ldots$ one computes the true exact values of $\phi(t)$ at positions $2^{-j}\,\ell$ $(0 \leq \ell \leq 2^j\,L, \ell$ odd) using (S)
- Note that all values computed by this method are exact, so they can be confidentially used to interpolate (if j is sufficiently big)

- Equation (S) taken for $t \in \{0, 1, 2, \ldots, L\}$ gives

$$\begin{aligned}
\phi(0) &= \sqrt{2} \cdot h_0\, \phi(0) \\
\phi(1) &= \sqrt{2} \cdot (h_2\, \phi(0) + h_1\, \phi(1) + h_0\, \phi(2)) \\
\phi(2) &= \sqrt{2} \cdot (h_4\, \phi(0) + h_3\, \phi(1) + h_2\, \phi(2) + h_1\, \phi(3) + h_0\, \phi(4)) \\
&\vdots \\
\phi(L-1) &= \sqrt{2} \cdot (h_L\, \phi(L-2) + h_{L-1}\, \phi(L-1) + h_{L-2}\, \phi(L)) \\
\phi(L) &= \sqrt{2} \cdot h_L\, \phi(L)
\end{aligned}$$

- The first and the last of these equations are easily satisfied by $\phi(0) = \phi(L) = 0$
- The remaining $L-1$ equations can be seen as an eigenvalue problem:

- The eigenvalue problem

$$\begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(L-2) \\ \phi(L-1) \end{bmatrix} = \sqrt{2} \cdot \begin{bmatrix} h_1 & h_0 & 0 & 0 & \dots & 0 \\ h_3 & h_2 & h_1 & h_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & h_L & h_{L-1} & h_{L-2} & h_{L-3} \\ 0 & \dots & 0 & 0 & h_L & h_{L-1} \end{bmatrix} \cdot \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(L-2) \\ \phi(L-1) \end{bmatrix}$$

- From this values $\phi(1), \phi(2), \dots, \phi(L-1)$ can be computed exactly
- Then one proceeds by dyadic interpolation from (S) for $j = 1, 2, \dots$:

$$\phi(2^{-j}\,\ell) = \sqrt{2} \sum_k h_k\, \phi(2^{-j+1}\,\ell - k) \quad (0 \leq \ell \leq 2^j\, L, \ell \text{ odd}).$$