Fourier Essentials

WTBV

November 18, 2015

key aspects of the Fourier transform

- Fundamental idea: functions/signals have a life both in time/space and in frequency domain — and both aspects are equivalent
- Motivation: Fourier transform can be obtained from Fourier series by a limiting process
- Basic properties of FT
  - Translation and Dilation — basic wavelet operations
  - Derivation — smoothness properties of wavelets
  - Convolution — filtering properties of wavelets
- Advanced properties of FT
  - Time/frequency localization, duality and uncertainty
  - Poisson's formula and sampling
- Fourier transform theory is
  - important — immense number of applications
  - not easy — making ideas rigorous requires lot of work
  - beautiful — leads into a new universe

- Some highlights
  - Trying to make Fourier's ideas precise spawned lots of new mathematics (convergence concepts, set theory, distributions,...)
  - The Cooley-Tukey 1965 paper on FFT is the most frequently cited article in all of mathematics
  - First US patent for a mathematical algorithm for a variant of FFT
  - About 3/4 of all Nobel prices in physics were awarded for work done with Fourier analysis
  - Other notable Nobel prices:
    Crick/Watson/Wilkins (1962): DNA structure by diffraction
    Cormack/Hounsfield (1979): computed tomography
    Hauptman/Karle (1985): structure of molecules by X-ray diffraction
    Lauterbur/Mansfield (2003): MRI
  - Other fields: PDE, Quantum Mechanics, Signals and Systems, Fourier Optics, ...

- A function $f : \mathbb{R} \to \mathbb{C}$ is p-periodic, if $f(t+p) = f(t)$ for all $t \in \mathbb{R}$.
- Equivalently:
  $f(t)$ is defined in any real interval $[a, b)$ of length $b - a = p$ and is extended periodically
- Simple 1-periodic funktions are the harmonics

$$\sin(2\pi kt) \;\; (k \geq 1), \quad \cos(2\pi kt) \;\; (k \geq 0), \quad \omega_k(t) = \mathrm{e}^{2\pi ikt} \;\; (k \in \mathbb{Z})$$

- Superposition principle: linear combinations of 1-periodic functions are again 1-periodic functions
- Fourier's Idea (1807): "Any" 1-periodic function $f(t)$ can be represented as a superposition (*Fourier series*) of harmonics, i.e., there are sequences $(a_k)_{k\geq 0}, (b_k)_{k\geq 1}, (c_k)_{k\in\mathbb{Z}} \in \mathbb{C}$ s.th.

$$\begin{aligned} f(t) \text{ ``=''} \;& \frac{a_0}{2} + \sum_{k>0} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \\ \text{``=''} \;& \sum_{k\in\mathbb{Z}} c_k \, \mathrm{e}^{2\pi ikt} \end{aligned}$$

- $\mathcal{L}^2([0,1))$: Hilbert space of square-integrable (in the sense of Lebesgue) functions $f : [0,1) \to \mathbb{C}$ with inner product

$$\langle\, f \,|\, g \,\rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt < \infty$$

and norm

$$\|f\|^2 = \langle\, f \,|\, f \,\rangle = \int_0^1 |f(t)|^2\,dt < \infty$$

- The families

$$\left\{\omega_k(t) = \mathrm{e}^{2\pi i k t}\right\}_{k\in\mathbb{Z}}$$

and

$$\{\sin(2\pi k t)\}_{k\geq 1} \cup \{\cos(2\pi k t)\}_{k\geq 0}$$

are orthonormal families (even Hilbert bases) of $\mathcal{L}^2([0,1))$:

$$\langle\, \omega_k \,|\, \omega_\ell \,\rangle = \int_0^1 \mathrm{e}^{2\pi i(k-\ell)t}\,dt = \delta_{k,\ell}$$

- Similarly for the family of harmonics sin-cos

- Fourier coefficients (*Analysis*)

$$c_k = \widehat{f}[k] = \langle\, f \,|\, \omega_k \,\rangle = \int_0^1 f(t)\,\mathrm{e}^{-2\pi i k t}\,dt \quad (k \in \mathbb{Z})$$

- Fourier series (*Synthesis*)

$$f(t) = \sum_{k\in\mathbb{Z}} \langle\, f \,|\, \omega_k \,\rangle\, \omega_k(t) = \sum_{k\in\mathbb{Z}} \widehat{f}[k]\,\mathrm{e}^{2\pi i k t}$$

- For $f \in \mathcal{L}^2([0,1))$ one has

$$S_N(t) = \sum_{k=-N}^{N} \widehat{f}[k]\,\mathrm{e}^{2\pi i k t} \rightarrow_{N\rightarrow\infty} f(t)$$

  in the sense of $\mathcal{L}^2$-convergence (optimal $\mathcal{L}^2$-approximation)
- Stronger assertions about convergence are possible, but more difficult to obtain

- Important aspects
  - The Fourier coefficients of $f(t)$ depend on the behavior of $f(t)$ over the whole interval $[0, 1)$
  - The basis functions $\omega_k(t) = \mathrm{e}^{2\pi ikt}$ are
    - perfectly localized w.r.t. frequency
    - not at all localized w.r.t.time/space
  - The family $\left\{\omega_k(t) = \mathrm{e}^{2\pi ikt}\right\}_{k\in\mathbb{Z}}$ is a complete basis (Hilbert basis) in $\mathcal{L}^2([0, 1))$

- Hilbert space of sequences (“discrete signals with finite energy‘)
  - complex bi-infinite sequences

    $$\boldsymbol{x} = (\ldots, x[-1], x[0], x[1], x[2], \ldots) = (x[k])_{k\in\mathbb{Z}} \text{ with } x[k] \in \mathbb{C} \ (k \in \mathbb{Z})$$

  - the relevant vector space is ℓ^2

    $$\ell^2 = \left\{ \boldsymbol{x} = (x[k])_{k\in\mathbb{Z}} \,;\, \sum_{k\in\mathbb{Z}} |x[k]|^2 < \infty \right\}$$

    with inner product $\langle \boldsymbol{x} \mid \boldsymbol{y} \rangle = \sum_{k\in\mathbb{Z}} x[k] \cdot \overline{y[k]}$
    and norm $\|\boldsymbol{x}\|^2 = \sum_{k\in\mathbb{Z}} |x[k]|^2$

- *Parseval-Plancherel* property: The mapping

$$f(t) \longmapsto \left(\widehat{f}[k]\right)_{k\in\mathbb{Z}}$$

  - is a linear mapping $\mathcal{L}^2([0,1)) \longmapsto \ell^2$
  - is an *isometry*, which means

$$\langle\, f \mid g \,\rangle_{\mathcal{L}^2} = \int_0^1 f(t)\,\overline{g(t)}\,dt = \sum_{k\in\mathbb{Z}} \widehat{f}[k]\,\overline{\widehat{g}[k]} = \langle\, \widehat{f} \mid \widehat{g} \,\rangle_{\ell^2}$$

  - is surjective (the *Riesz-Fischer* theorem)
- Conclusion: $\mathcal{L}^2([0,1))$ and ℓ^2 are isomorphic as Hilbert spaces

- Everything carries over routinely from $[0,1)$ to arbitrary finite intervals $[a,b)$ and p-periodic functions with $p = b - a$
- $\mathcal{L}^2([a,b))$ has a basis of functions

$$\left\{\omega_k(t/p) = \mathrm{e}^{2\pi i k t/p}\right\}_{k\in\mathbb{Z}}$$

  and similarly for sin-cos
- the inner product in $\mathcal{L}^2([a,b))$ is

$$\langle\, f \,|\, g \,\rangle = \frac{1}{p}\int_a^b f(t)\,\overline{g(t)}\,dt$$

- Fourier coefficients (*Analysis*)

$$\widehat{f}[k] = \langle\, f \,|\, \omega_k(t/p)\,\rangle = \frac{1}{p}\int_a^b f(t)\,\mathrm{e}^{-2\pi i k t/p}\,dt$$

- Fourier series (*Synthesis*)

$$f(t) = \sum_{k\in\mathbb{Z}} \widehat{f}[k]\,\mathrm{e}^{2\pi i k t/p}$$

- The *Gibbs-Wilbraham* phenomenon
  - describes the convergence of the approximations $s_N(t)$ at a jump discontinuity of the function $f(t)$
  - typical example: $f(t)$ as extension of $\chi_{[-1/2,1/2)}(t)$ to a 2-periodic function
  - Fourier coefficients

$$\widehat{f}[k] = \frac{1}{2}\int_{-1}^{1} \chi_{[-1/2,1/2)}(t)\frac{\mathrm{e}^{-\pi ikt}}{\sqrt{2}}\,dt = \begin{cases} 1/2 & k = 0 \\ 0 & k \neq 0 \text{ and even} \\ \frac{(-1)^{(k-1)/2}}{\pi k} & k \text{ odd} \end{cases}$$

  - Fourier series

$$\frac{1}{2} + \frac{2}{\pi}\cos(\pi t) - \frac{2}{3\pi}\cos(3\pi t) + \frac{2}{5\pi}\cos(5\pi t) \mp \cdots$$

- Approximation

$$S_N(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{N}(-1)^{n-1}\frac{\cos((2n-1)\pi t)}{2n-1}$$

- graphical display

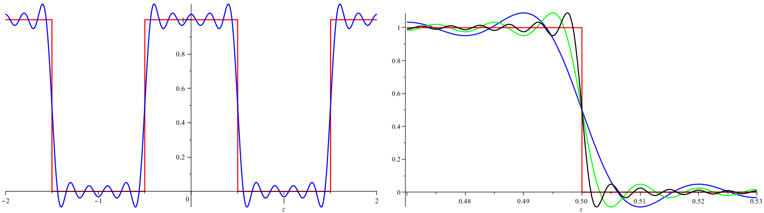


Figure: S_5 (left), S_{50}, S_{100}, S_{200} (right)

- Notabene: the "overshooting" of the approximation does NOT disappear as $N \to \infty$!

- Consider a function $f(t)$ which vanishes outside a finite interval $[-L_0, L_0)$, and for $L \geq L_0$ consider

$$f_L(t) = \begin{cases} f(t) & \text{for } |t| \leq L_0 \\ 0 & \text{for } L_0 \leq |t| \leq L \end{cases}$$

  as a 2L-periodic function
- Fourier coefficients (analysis)

$$\widehat{f_L}[k] = \frac{1}{2L} \int_{-L}^{L} f_L(t)\, \mathrm{e}^{-2\pi i k t/2L}\, dt$$

- Synthesis formula

$$f_L(t) = \sum_{k \in \mathbb{Z}} \widehat{f_L}[k]\, \mathrm{e}^{2\pi i k t/2L}$$

- Now define for all $s \in \mathbb{R}$ and $L \geq L_0$

$$\widehat{f}(s) = \int_{-L}^{L} f_L(t)\, \mathrm{e}^{-2\pi i\, s\, t}\, dt$$

This definition is independent of L !

- Then for all $s \in \mathbb{R}$ of the form $s = \frac{k}{2L}$ with $k \in \mathbb{Z}$ it is true that

$$\widehat{f}(s) = 2L \cdot \widehat{f_L}[k]$$

- For $L \geq L_0$ one has

$$g_L : \frac{k}{2L} \longmapsto 2L \cdot \widehat{f_L}[k] \quad (= \widehat{f}(\frac{k}{2L})) \qquad (k \in \mathbb{Z})$$

as a discrete function

- Conclusion: For $L \to \infty$ the graphs of the discrete functions g_L "converge" to the graph of a function $s \longmapsto \widehat{f}(s)$ defined on $\mathbb{R}$

- Furthermore

$$f_L(t) = \sum_{k\in\mathbb{Z}} \widehat{f_L}[k]\, \mathrm{e}^{2\pi ikt/2L} = \sum_{k\in\mathbb{Z}} \frac{1}{2L}\, \widehat{f}(\frac{k}{2L})\, \mathrm{e}^{2\pi i\,(k/2L)\,t}$$

The right-hand side is the Riemann sum for the integral

$$\int_{\mathbb{R}} \widehat{f}(s)\, \mathrm{e}^{2\pi ist}\, ds$$

- Thus for $L \to \infty$ on expects a synthesis formula

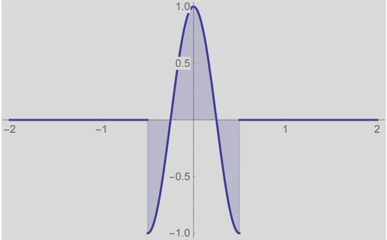
$$f(t) := f_\infty(t) = \int_{\mathbb{R}} \widehat{f}(s)\, \mathrm{e}^{2\pi ist}\, ds$$

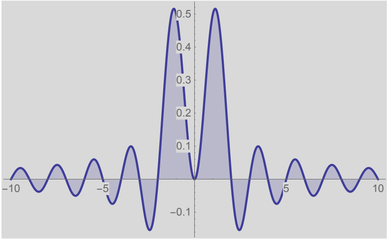
- together with an analysis formula

$$\widehat{f}(s) = \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi ist}\, dt$$

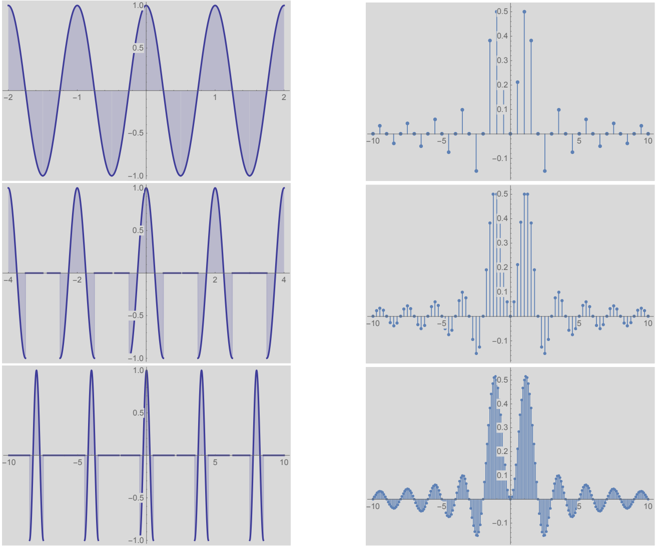
- Example: A 1-periodic function and it Fourier transform

$$f(t) = \cos(2\pi t) \quad |t| \leq 1/2 \qquad \widehat{f}(s) = \frac{s \sin \pi s}{\pi - \pi s^2}$$





- Schematic display of the transition Fourier series → Fourier transform for $f(t) = \cos(2\pi t)$ with $|t| \leq 1/2$ and $L = 1, 2, 4$



- The relevant Hilbert space for the Fourier transform is $\mathcal{L}^2(\mathbb{R})$, the vector space of (Lebesgue-)square-integrable functions on $\mathbb{R}$
- Inner product and norm in $\mathcal{L}^2(\mathbb{R})$

$$\langle f \,|\, g \rangle = \int_{\mathbb{R}} f(t)\,\overline{g(t)}\,dt \qquad \|f\|^2 = \langle f \,|\, f \rangle = \int_{\mathbb{R}} |f(t)|^2\,dt$$

- Serious defect: simple functions like polynomials, trigonometric functions and complex exponentials do NOT belong to $\mathcal{L}^2(\mathbb{R})$; in particular, the family

$$\left\{\omega_s(t) = \mathrm{e}^{2\pi i s t}\right\}_{s\in\mathbb{R}}$$

cannot be a basis of the Hilbert space $\mathcal{L}^2(\mathbb{R})$!

- For integrable functions $f(t)$ the inner products

$$\widehat{f}(s) := \langle f \,|\, \omega_s(t) \rangle = \int_{\mathbb{R}} f(t)\,\mathrm{e}^{-2\pi i s t}\,dt$$

are nevertheless well defined!

- Definition: For "suitable" functions $f : \mathbb{R} \to \mathbb{C}$ their *Fourier transform* $\widehat{f} : \mathbb{R} \to \mathbb{C}$ is defined by

$$\text{(Analysis)} \qquad \widehat{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi ist} dt \quad (s \in \mathbb{R})$$

  Often denoted as $\mathcal{F}_t[f(t)]$ or $\mathcal{F}[f]$ instead of $\widehat{f}$

- *Inversion formula*: If the function $f(t)$ is sufficiently well-behaved, one expects that it can be reconstructed from its Fourier transform $\widehat{f}$ by:

$$\text{(Synthesis)} \qquad f(t) = \int_{-\infty}^{\infty} \widehat{f}(s)\, e^{2\pi ist} ds \quad (t \in \mathbb{R})$$

  Often denoted as $f = \mathcal{F}_s^{-1}[\widehat{f}(s)$ or $f = \mathcal{F}^{-1}[\widehat{f}]$

- If this holds, then $f(t)$ is "continuous linear combination" (superposition) of harmonics (complex exponentials)
- $\widehat{f}(s)$ is the amplitude or intensity of $\omega_s(t) = \mathrm{e}^{2\pi i\,s\,t}$ in $f(t)$

- Attention! In the literature there are many slightly different conventions used of the definition of the *Fourier transform*. The type of expressions used is

$$\widehat{f}(s) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(t)\, e^{i\, b\, s\, t} dt$$

- with the following conventions
  - $(a, b) = (0, 1)$ (modern physics, `Mathematica`)
  - $(a, b) = (1, -1)$ (mathematics, systems theory, `Maple`)
  - $(a, b) = (-1, 1)$ (classical physics)
  - $(a, b) = (0, -2\pi)$ (signal processing, *this lecture*)
- The formula for the inverse transform has to be adapted accordingly

- Comments:
  - Fourier transform is a linear transformation, it is even *unitary* (=complex-orthogonal) transform ($\rightarrow$ Parseval-Plancherel)
  - The definition of the Fourier transform makes sense if $f \in \mathcal{L}^1(\mathbb{R})$, i.e., if f is integrable (in the sense of Lebesgue): $\|f\|_1 = \int_{\mathbb{R}} |f(t)|\,dt < \infty$
  - For the inversion formula to make sense, one should have $\widehat{f} \in \mathcal{L}^1(\mathbb{R})$, which unfortunately is not guaranteed,

    it holds, however, e.g., if $f \in \mathcal{L}^1(\mathbb{R})$ is continuous
  - The complex exponentials $t \mapsto \mathrm{e}^{2\pi i s t}$ belong neither to $\mathcal{L}^1(\mathbb{R})$, nor to $\mathcal{L}^2(\mathbb{R})$, i.e., they cannot be taken as basis functions
  - In order to get a satisfactory theory of the Fourier transform one has to extend the space of admissible functions ($\rightarrow$ *distributions*)

- Examples (1)

$f(t)$	$\leftrightsquigarrow$	$\widehat{f}(s)$
$\chi_{[-a,a]}(t)$	$\leftrightsquigarrow$	$\dfrac{\sin(2\pi as)}{\pi s}$
$(1 - \left[\frac{t}{a}\right]) \cdot \chi_{[-a,a]}(t)$	$\leftrightsquigarrow$	$\dfrac{\sin^2(\pi as)}{a(\pi s)^2}$
$\mathrm{e}^{-a\,\lvert t\rvert}$	$\leftrightsquigarrow$	$\dfrac{2a}{a^2 + (2\pi s)^2}$
$\mathrm{e}^{-a\,t^2}$	$\leftrightsquigarrow$	$\sqrt{\dfrac{\pi}{a}}\,\mathrm{e}^{-(\pi s)^2/a}$

- Fourier transform can/must be extended to cover familiar functions
- Examples (2)

$f(t)$	$\leftrightsquigarrow$	$\widehat{f}(s)$
1	$\leftrightsquigarrow$	$\delta(s)$
$\mathrm{e}^{2\pi i\, at}$	$\leftrightsquigarrow$	$\delta(a-s)$
$1 + a\,t + b\,t^2$	$\leftrightsquigarrow$	$\delta(s) + \frac{ia\,\delta'(s)}{2\pi} - \frac{b\,\delta''(s)}{4\pi^2}$
$\frac{1}{1+a\,t^2}$	$\leftrightsquigarrow$	$\frac{\pi}{\sqrt{a}}\left(\mathrm{e}^{2\pi s/\sqrt{a}}\,\theta(-2\pi s) + \mathrm{e}^{-2\pi s/\sqrt{a}}\,\theta(2\pi s)\right)$

where

- $\theta(t) = \chi_{t>0}(t)$ denotes Heaviside's jump function
- $\delta(t) = \frac{d}{dt}\,\theta(t)$ denotes Diracs Delta-"function"

- A possible definition of the *distribution* $\delta(t)$ is furnished by

$$\delta : f(t) \mapsto f(0)$$

for "sufficiently well-behaved" functions $f(t)$ ("test functions"), often written as

$$\int_{-\infty}^{\infty} \delta(t)\, f(t)\, dt = f(0)$$

- No function in the traditional sense can have this property, so $\delta(t)$ is not a function, but a linear functional

- Translation vs. Modulation

$$\widehat{f(t-a)}(s) = e^{-2\pi i a s} \cdot \widehat{f(t)}(s)$$

- Dilation (Scaling)

$$\sqrt{a}\,\widehat{f(a\,t)}(s) = \frac{1}{\sqrt{a}}\,\widehat{f(t)}\left(\frac{s}{a}\right)$$

- Derivation vs. Multiplication

$$\widehat{\frac{d}{dt}f(t)}(s) = 2\pi i s \cdot \widehat{f(t)}(s)$$

- Convolution

$$(f \star g)(t) := \int_{-\infty}^{\infty} f(x)\,g(t-x)\,dx$$

Convolution theorem

$$\widehat{(f \star g)(t)}(s) = \widehat{f(t)}(s) \cdot \widehat{g(t)}(s)$$

- Dilation
  - The *a-Dilation* $(D_a f)(t)$ of a function $f(t)$ is defined as

$$(D_a f)(t) = \sqrt{a}\, f(a\, t)$$

  - Dilation means *stretching* (for $0 < a < 1$) resp. *squeezing* (for $a > 1$) of the graph of f so that the norm is conserved

$$\|D_a f\| = \|f\|$$

  - The behavior of the Fourier transform w.r.t. dilation can be succinctly described by

$$\widehat{D_a f} = D_{1/a} \widehat{f}$$

    This antagonistic property is one of the characteristics of the Fourier transform ($\rightarrow$ uncertainty relation)

- Derivation vs. multiplication
  Under suitable conditions on $f(t)$ by partial integration or by interchanging integration and derivation:
  - $\widehat{f'(t)}(s) = (2\pi i s) \cdot \widehat{f}(s)$

$$\begin{aligned}\widehat{f'(t)}(s) &= \int_{\mathbb{R}} f'(t)\, \mathrm{e}^{-2\pi i s t}\, dt \\ &= \mathrm{e}^{-2\pi i s t}\, f(t)\,\Big|_{t\to-\infty}^{t\to+\infty} + (2\pi i s) \cdot \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi i s t}\, dt \\ &= (2\pi i s) \cdot \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi i s t}\, dt\end{aligned}$$

  - $\widehat{t \cdot f(t)}(s) = \frac{-1}{2\pi i} \cdot \frac{d}{ds} \widehat{f}(s)$

$$\widehat{t \cdot f(t)}(s) = \int_{\mathbb{R}} t \cdot f(t)\, \mathrm{e}^{-2\pi i s t}\, dt = \frac{-1}{2\pi i} \cdot \frac{d}{ds} \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi i s t}\, dt$$

- Derivation: Smoothness and vanishing at infinity
  - *Riemann-Lebesgue* Lemma

$$f \in \mathcal{L}^1(\mathbb{R}) \quad \Rightarrow \quad \begin{cases} \widehat{f} \text{ is uniformly continuous on } \mathbb{R} \\ \text{and } \lim_{|s|\to\infty} \widehat{f}(s) = 0 \end{cases}$$

  - "$t^N \cdot f(t) \in \mathcal{L}^1(\mathbb{R})$" means: $f(t)$ vanishes fast as $t \to \pm\infty$:

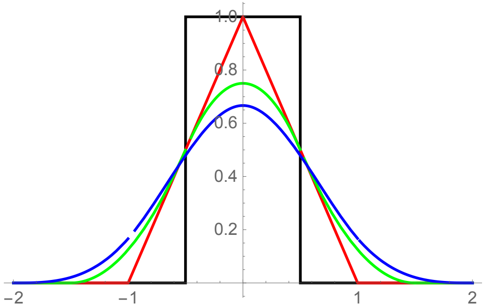
$$\int_{\mathbb{R}} |t^N f(t)|\, dt < \infty, \quad \text{so typically} \quad f(t) \in \mathcal{O}(t^{-N-1-\varepsilon})$$

  - The faster a function $f(t)$ vanishes as $t \to \pm\infty$, the smoother (higher order differentiable) is $\widehat{f}(s)$ – and conversely

$$\left.\begin{matrix} f(t) \in \mathcal{L}^1(\mathbb{R}) \\ t^N \cdot f(t) \in \mathcal{L}^1(\mathbb{R}) \end{matrix}\right\} \Leftrightarrow \begin{cases} \widehat{f} \in \mathcal{C}^N(\mathbb{R}) \quad \text{and} \\ \frac{d^k}{ds^k}\widehat{f}(s) = \frac{-1}{(2\pi i)^k}\, \widehat{t^k f(t)}(s) \quad (0 \le k \le N) \end{cases}$$

  - *Derivation* and *multiplication with the variable* are "complementary"

- B-Spline functions and their Fourier transforms

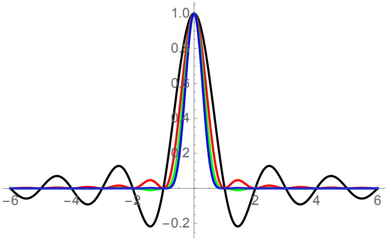


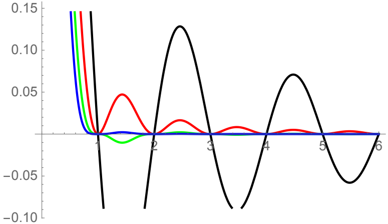
Iterated convolutions of the box function $b(t)$

$$b^{\star n}(t) = (b \star b \star \cdots \star b)(t) \quad (n \text{ factors})$$

$b^{\star n}$ is $(n-2)$-fold differentiable
n: 1=black, 2=red, 3=green, 4 = blue

- B-Spline functions and their Fourier transforms





The Fourier transforms are the functions

$$\widehat{b^{\star n}}(s) = \mathrm{sinc}(\pi s)^n = \frac{\sin(\pi s)^n}{(\pi s)^n} \in \mathcal{O}(s^{-n})$$

n: 1=black, 2=red, 3=green, 4 = blue

- Definition of convolution

$$(f \star g)(t) := \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx$$

- If $g(t) = \omega_s(t) = \mathrm{e}^{2\pi ist}$:

$$\begin{aligned}(f \star \omega_s)(t) &= \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{2\pi i\, s(t-x)}\, dx \\ &= \mathrm{e}^{2\pi i\, st} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi isx} = \widehat{f}(s) \cdot \omega_s(t)\end{aligned}$$

- Convolution by a fixed function $f(t)$

$$\mathcal{C}_f : g(t) \mapsto (f \star g)(t)$$

  is a linear transformation which has
  - the complex exponentials $\omega_s(t) = \mathrm{e}^{2\pi i\, st}$ as *eigenfunctions*
  - with Fourier transform value $\widehat{f}(s)$ as the corresponding *eigenvalues*
- Convolution with $\delta(t)$ replicates $f(t)$

$$(\delta \star f)(t) = \int_{-\infty}^{\infty} \delta(x) f(t-x)\, dx = f(t)$$

- The convolution theorem

$$\begin{array}{ccc} f, g & \stackrel{\mathcal{F}}{\Longrightarrow} & \widehat{f}, \widehat{g} \\ \Downarrow \star & & \Downarrow \cdot \\ f \star g & \stackrel{\mathcal{F}}{\Longrightarrow} & \widehat{f \star g} = \widehat{f} \cdot \widehat{g} \end{array}$$

- Main application of convolution

"Filtering in the frequency domain"

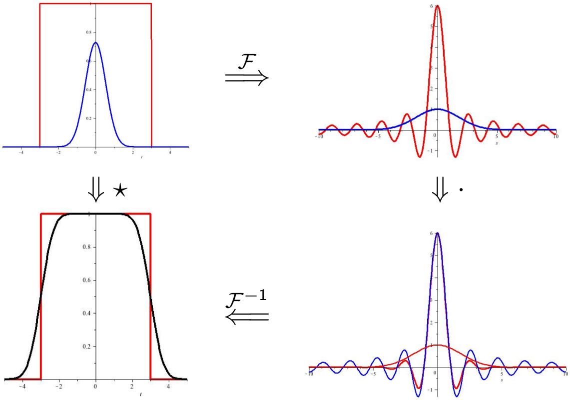
$$\begin{array}{ccc} f, g & \stackrel{\mathcal{F}}{\Longrightarrow} & \widehat{f}, \widehat{g} \\ \Downarrow \star & & \Downarrow \cdot \\ f \star g = \mathcal{F}^{-1}(\widehat{f} \cdot \widehat{g}) & \stackrel{\mathcal{F}^{-1}}{\Longleftarrow} & \widehat{f} \cdot \widehat{g} \end{array}$$

- Proof of the convolution theorem (sketch)

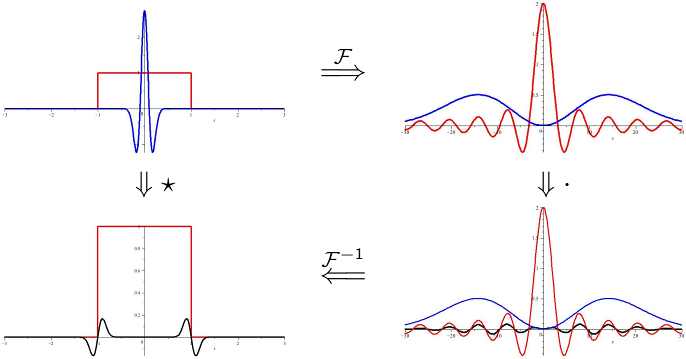
$$
\begin{aligned}
\widehat{f \star g}(s) &= \int_{\mathbb{R}} (f \star g)(t)\, \mathrm{e}^{-2\pi i s t}\, dt && \text{def. of FT} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\, g(t-x)\, dx\, \mathrm{e}^{-2\pi i s t}\, dt && \text{def. of } \star \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i s x} g(t-x)\, \mathrm{e}^{-2\pi i s (t-x)}\, dx\, dt \\
&= \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i s x} \int_{\mathbb{R}} g(t-x)\, \mathrm{e}^{-2\pi i s (t-x)}\, dt\, dx && \int_t \int_x \equiv \int_x \int_t \\
&= \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i s x} \int_{\mathbb{R}} g(t)\, \mathrm{e}^{-2\pi i s t}\, dt\, dx && t \mapsto t + x \\
&= \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i x}\, \widehat{g}(s)\, dx && \text{def. of } FT \\
&= \widehat{f}(s) \cdot \widehat{g}(s) && \text{def. of } FT
\end{aligned}
$$

The crucial point is the change of the order of integration!

- Low-pass filtering with a Gauss filter



- High-pass filtering with a Mexhat filter



- A fundamental consequence of the convolution theorem:

  The *Parseval-Plancherel identity*: Fourier transform is an isometry!
- For $f, g \in \mathcal{L}^2$ s.th. also $\widehat{f}, \widehat{g} \in \mathcal{L}^2$, one has

$$\langle\, f \,|\, g \,\rangle = \langle\, \widehat{f} \,|\, \widehat{g} \,\rangle \quad \text{and in particular} \quad \begin{cases} \|f\| = \|\widehat{f}\| \\ f \perp g \quad \Leftrightarrow \quad \widehat{f} \perp \widehat{g} \end{cases}$$

Sketch of proof
Define $\widetilde{g}(t) = \overline{g(-t)}$ and check that $\widehat{\widetilde{g}}(s) = \overline{\widehat{g}(s)}$, then

$$\langle\, \widehat{f} \,|\, \widehat{g} \,\rangle = \int \widehat{f}(s) \cdot \overline{\widehat{g}(s)}\, ds = \int \widehat{f}(s) \cdot \widehat{\widetilde{g}}(s)\, ds =$$

$$\int \widehat{(f \star \widetilde{g})}(s)\, ds = (f \star \widetilde{g})(0) = \int f(t) \cdot \widetilde{g}(-t)\, dt$$

$$= \int f(t) \cdot \overline{g(t)}\, dt = \langle\, f \,|\, g \,\rangle$$

- Uncertainty relation
  - For $f(t) \in \mathcal{L}^2(\mathbb{R})$ with

$$\|f\|^2 = \int_{\mathbb{R}} |f(t)|^2 \, dt = 1$$

then $t \mapsto |f(t)|^2$ can be seen as a probability density function on $\mathbb{R}$

  - Expectation and variance of this probability density are given by

$$\mu(f) = \int_{\mathbb{R}} t \, |f(t)|^2 \, dt \qquad \sigma^2(f) = \int_{\mathbb{R}} (t - \mu(f))^2 \, |f(t)|^2 \, dt$$

  - Because of the Parseval-Plancherel identity one also has $\|\widehat{f}\| = 1$; $\mu(\widehat{f})$ and $\sigma^2(\widehat{f})$ are defined analogously
  - Then the *Heisenberg inequality* holds:

$$\sigma^2(f) \cdot \sigma^2(\widehat{f}) \geq \frac{1}{(4\pi)^2}$$

(For a proof see the Lecture Notes)

- Examples

$f(t,a)$	$\sigma^2(f)$	$\widehat{f}(s,a)$	$\sigma^2(\widehat{f})$
$\sqrt{a}\,\chi_{[-1/2,1/2]}(at)$	$\frac{1}{12a^2}$	$\sqrt{a}\,\frac{\sin(\pi\,s/a)}{\pi\,s}$	∞
$\sqrt{\frac{3}{2a}}\,(1-\lvert at\rvert)\cdot\chi_{-1/a,1/a}(t)$	$\frac{1}{10a^2}$	$\sqrt{\frac{3}{2a}}\,\frac{(\sin(\pi\,s/a))^2}{\pi^2 s^2}$	$\frac{3a^2}{4\pi^2}$
$\sqrt{a}\,\mathrm{e}^{-a\lvert t\rvert}$	$\frac{1}{2a^2}$	$2\,\frac{a^{3/2}}{a^2+4\,\pi^2 s^2}$	$\frac{a^2}{4\pi^2}$
$\sqrt[4]{\frac{2a}{\pi}}\,\mathrm{e}^{-a\,t^2}$	$\frac{1}{4a}$	$\sqrt[4]{\frac{2a}{\pi}}\,\mathrm{e}^{-\frac{\pi^2 s^2}{a}}$	$\frac{a}{4\pi^2}$

- Graphical illustration of uncertainty: *Heisenberg boxes*
  - For any function $f(t)$ and $a > 0, b \in \mathbb{R}$ let

$$f_{a,b}(t) = \sqrt{a} \cdot f(a\,t - b), \quad \mu_{a,b} = \mu(f_{a,b}), \quad \sigma^2_{a,b} = \sigma^2(f_{a,b}),$$

    and similarly for $\widehat{f}(s)$
  - Then

$$\mu_{a,b} = \frac{\mu + b}{a}, \quad \sigma^2_{a,b} = \frac{\sigma^2}{a^2}, \quad \widehat{\mu}_{a,b} = a\,\widehat{\mu}, \quad \widehat{\sigma}^2_{a,b} = a^2\,\widehat{\sigma}^2$$

  - The Heisenberg box for the function $f(t)$ is the rectangle in the (s,t)-plane centered at $(\mu_{a,b}, \widehat{\mu}_{a,b})$ and with side lengths $(\sigma_{a,b}, \widehat{\sigma}_{a,b})$. This box characterizes the simultaneous uncertainty of $f(t)$ in the time/space domain and in the frequency domain
    The box area $\sigma_{a,b} \cdot \widehat{\sigma}_{a,b} \geq \frac{1}{4\pi}$ is independent of scaling a and translation b !

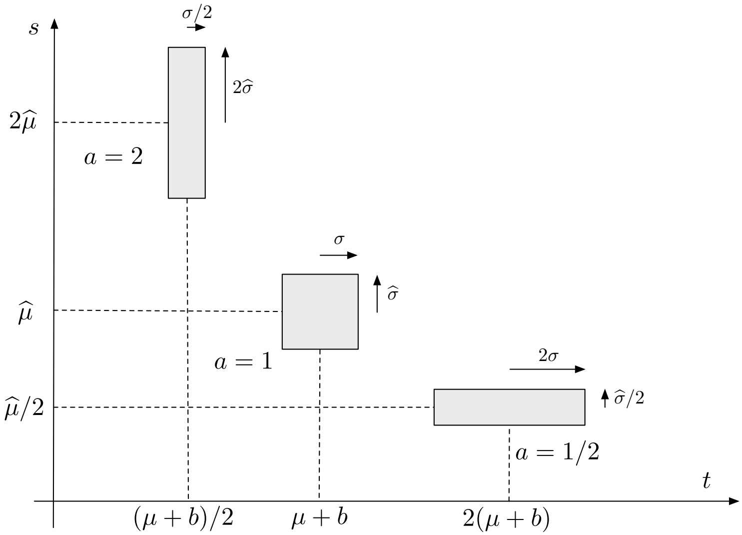


Figure: Heisenberg boxes for $f_{a,b}(t)$ with $a = 1/2, a = 1$ and $a = 2$

- *Poisson's formula*

  For any sufficently well-behaved function $f : \mathbb{R} \to \mathbb{C}$ there is a relation
  - between the values $f(k)$ $(k \in \mathbb{Z})$ at integer arguments
  - and the values $\widehat{f}(s-n)$ $(n \in \mathbb{Z})$ of its Fourier transform

$$\sum_{n=-\infty}^{\infty} \widehat{f}(s-n) = \sum_{k=-\infty}^{\infty} f(k)\, \mathrm{e}^{-2\pi isk} \quad (s \in \mathbb{R})$$

- Note: the sum on the l.h.s. defines a 1-periodic function, the sum on the r.h.s. is a Fourier series
- In particular (take $s = 0$)

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{k=-\infty}^{\infty} f(k)$$

- Equivalent version of Poisson's formula (for $a > 0$)

$$\sum_{n=-\infty}^{\infty} f(t - n/a) = a \cdot \sum_{k=-\infty}^{\infty} \widehat{f}(k \cdot a)\, \mathrm{e}^{2\pi i\, t\, k\, a}$$

- Sketch of proof (case $a = 1$ suffices):
  $\phi(t) = \sum_n f(t-n)$ is 1-periodic,

  so if it has a Fourier series $\phi(t) = \sum_{k\in\mathbb{Z}} \varphi[k]\, \mathrm{e}^{2\pi ikt}$, then

$$\varphi[k] = \int_0^1 \phi(t)\, \mathrm{e}^{-2\pi ikt} dt = \sum_{n\in\mathbb{Z}} \int_0^1 f(t-n)\, \mathrm{e}^{-2\pi ikt} dt$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(t)\, \mathrm{e}^{-2\pi ikt} dt = \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-2\pi ikt} dt = \widehat{f}(k)$$

- *Shannon-Nyquist* sampling theorem

  If a signal $f : \mathbb{R} \to \mathbb{C}$ is *band-limited* in the sense that

$$|s| > \frac{1}{2a} \implies \widehat{f}(s) = 0,$$

then $f(t)$ can be perfectly reconstructed from its discrete *sampling values* $f(k \cdot a)$ $(k \in \mathbb{Z})$ by

$$\begin{aligned} f(t) &= \sum_{k \in \mathbb{Z}} f(k \cdot a) \cdot \frac{\sin(\frac{\pi}{a}(t - k \cdot a))}{\frac{\pi}{a}(t - k \cdot a)} \\ &= \sum_{k \in \mathbb{Z}} f(k \cdot a) \cdot \operatorname{sinc}\left(\frac{1}{a}(t - k \cdot a)\right) \end{aligned}$$

This is Shannon's formula $\quad (\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x})$

- From the band-limiting condition, only the $(n = 0)$-term from

$$\sum_{n\in\mathbb{Z}} \widehat{f}(s - n/a)$$

in Poisson's formula survives, so that

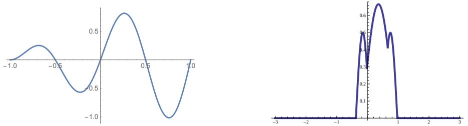
$$\widehat{f}(s) = a \sum_{k\in\mathbb{Z}} f(k\cdot a)\cdot e^{-2\pi is\, k\cdot a}$$

and thus

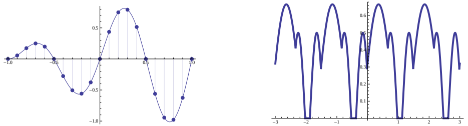
$$\begin{aligned} f(t) &= \int_{\mathbb{R}} \widehat{f}(s)\, e^{2\pi ist}\, ds \\ &= \int_{-1/(2a)}^{1/(2a)} \widehat{f}(s)\, e^{2\pi ist}\, ds \\ &= a \sum_{k\in\mathbb{Z}} f(k\cdot a) \int_{-1/(2a)}^{1/(2a)} e^{2\pi is(t-k\cdot a)} ds \\ &= \sum_{k\in\mathbb{Z}} f(k\cdot a)\cdot \frac{\sin\frac{\pi}{a}(t-k\cdot a)}{\frac{\pi}{a}(t-k\cdot a)} \end{aligned}$$

- What sampling really means?
  - Sampling a continuous signal with frequency a means: <u>repeating its spectrum periodically with distance a</u>

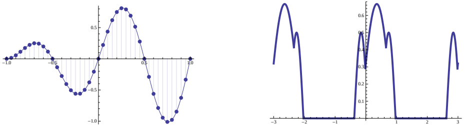
a function and its spectrum



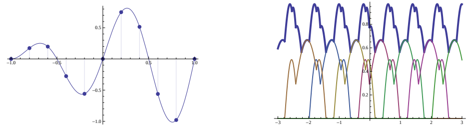
sampling with limit frequency



oversampling



undersampling



- In a purely formal way:

$$\delta(s) = \widehat{1}(s) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi ist}\,dt$$

The integral doesn't make sense, but ...

- ... if δ appears under an integral, it may work

$$\int_{\mathbb{R}} f(t)\,\delta(t)\,dt = \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} \mathrm{e}^{-2\pi ist}\,ds\,dt =$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t)\,\mathrm{e}^{-2\pi ist}\,dt\,ds = \int_{\mathbb{R}} \widehat{f}(s)\,ds = f(0)$$

which motivates the common definition (given earlier)

- Another characteristic property: $\delta \star f = f$

$$(\delta \star f)(t) = \int_{\mathbb{R}} \delta(s) \cdot f(t-s)\,ds = f(t)$$

i.e., $\delta(t)$ acts as a <u>neutral element w.r.t. convolution</u>
No "proper" function can have this property. Therefore

$$\widehat{f}(s) = \widehat{(\delta \star f)}(s) = \widehat{\delta}(s) \cdot \widehat{f}(s) \implies \widehat{\delta} \equiv 1$$

- Translation of δ
  - definition

$$\delta_a(t) = \delta(t-a) \quad \text{or} \quad \int f(t)\,\delta_a(t)\,dt = f(a) \quad \text{or} \quad \delta_a : f(t) \mapsto f(a)$$

  - multiplication with a function

$$f(t) \cdot \delta_a(t) \equiv f(a) \cdot \delta_a(t)$$

  - convolution with δ_a is translation

$$(f \star \delta_a)(t) = \int f(t-x)\,\delta_a(x)\,dx = f(t-a)$$

  - the Fourier transform of δ_a is

$$\widehat{\delta_a}(s) = \mathrm{e}^{-2\pi i a s}$$

- DIRAC's comb
  - definition

$$\text{Ш}(t) = \sum_{k\in\mathbb{Z}} \delta_k(t)$$

  - multiplication (the sampling property)

$$f(t) \cdot \text{Ш}(t) = \sum_{k\in\mathbb{Z}} f(t)\,\delta_k(t) = \sum_{k\in\mathbb{Z}} f(k)\,\delta_k(t)$$

  - convolution (the periodizing property)

$$(f \star \text{Ш})(t) = \sum_{k\in\mathbb{Z}} (f \star \delta_k)(t) = \sum_{k\in\mathbb{Z}} f(t-k)$$

  - the Fourier transform of $\text{Ш}(t)$ is

$$\widehat{\text{Ш}}(s) = \text{Ш}(s)$$

- DIRAC's comb and POISSON's formula

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \widehat{f}(t-k) &= (\widehat{f} \star Ш)(t) && \text{Ш-convolution} \\
&= (\widehat{f} \star \widehat{Ш})(t) && \text{FT of Ш} \\
&= (\widehat{f \cdot Ш})(t) && \text{convolution theorem} \\
&= (\widehat{\sum_{k\in\mathbb{Z}} f(k)\,\delta_k})(t) && \text{definition of Ш} \\
&= \sum_{k\in\mathbb{Z}} f(k)\,\widehat{\delta_k}(t) && \text{linearity of FT} \\
&= \sum_{k\in\mathbb{Z}} f(k)\,e^{-2\pi ikt} && \text{FT of } \delta_k
\end{aligned}
$$

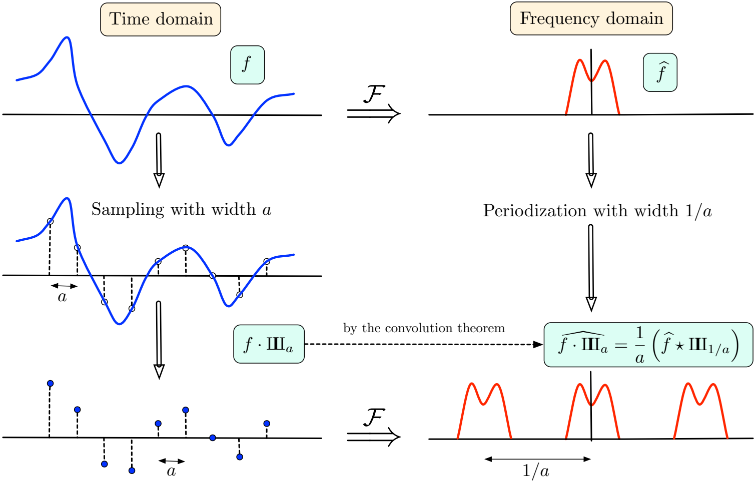


Figure: The sampling scheme

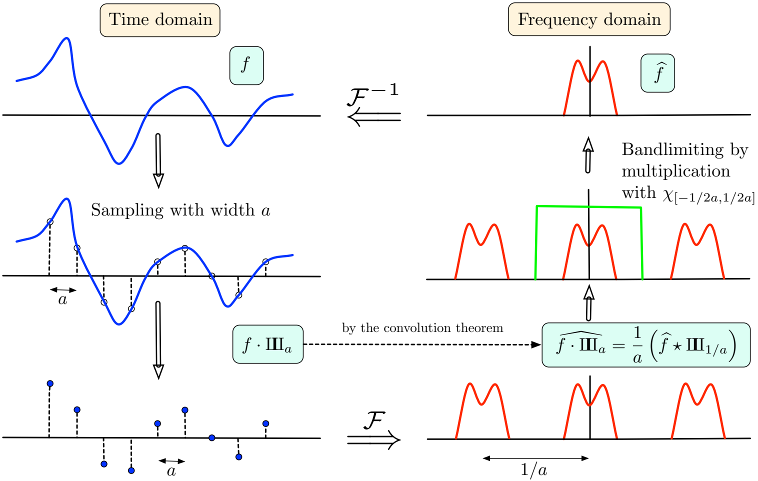


Figure: Reconstructing a sampled bandlimited signal

- $\delta_a(t) : f(t) \longmapsto f(a)$
- $\text{III}_a(t) = \sum_{k\in\mathbb{Z}} \delta_{k\cdot a}(t) = \frac{1}{a}\sum_{k\in\mathbb{Z}} e^{2\pi ikt/a}$

	$\delta_a(t)$	$\text{III}_a(t)$
action on $f(t)$	$f(a)$	$\sum_{k\in\mathbb{Z}} f(k\cdot a)$
product with $f(t)$	$f(a)\cdot\delta_a(t)$	$\sum_{k\in\mathbb{Z}} f(k\cdot a)\cdot\delta_{k\cdot a}(t)$
scaling with $p>0$	$\frac{1}{p}\,\delta_{a/p}(t)$	$\frac{1}{p}\,\text{III}_{a/p}(t)$
convolution with $f(t)$	$f(t-a)$	$\sum_{k\in\mathbb{Z}} f(t-k\cdot a)$
Fourier transform	$e^{-2\pi ias}$	$\frac{1}{a}\,\text{III}_{1/a}(s)$

- periodizing a function
  $f(t) \longmapsto \sum_{k\in\mathbb{Z}} f(t-k\cdot a) = (f\star\text{III}_a)(t)$
- sampling a function $f(t) \longmapsto \sum_{k\in\mathbb{Z}} f(k\cdot a)\cdot\delta_{k\cdot a}(t) = (f\cdot\text{III}_a)(t)$

$f(t)$ a b-bandlimited function
$\Rightarrow$ the copies of $\widehat{f}(s)$ contained in $\widehat{f \cdot \text{Ш}_{1/b}}$ do not overlap
$\Rightarrow$ $\widehat{f}$ can be recovered by

$$\widehat{f} = \Pi_b \cdot \widehat{f \cdot \text{Ш}_{1/b}} = b \cdot \Pi_b \cdot (\widehat{f} \star \text{Ш}_b)$$

where $\Pi_b(t) = \chi_{[-b/2,b/2]}(t)$. Now compute:

$$\begin{aligned} f &= \mathcal{F}^{-1}(b \cdot \Pi_b \cdot (\mathcal{F}(f) \star \text{Ш}_b)) && \text{inverse Fourier transform} \\ &= b \cdot \mathcal{F}^{-1}(\Pi_b) \star \mathcal{F}^{-1}(\mathcal{F}(f) \star \text{Ш}_b) && \text{convolution theorem} \\ &= b \cdot \mathcal{F}^{-1}(\Pi_b) \star (f \cdot \mathcal{F}^{-1}(\text{Ш}_b)) && \text{convolution theorem} \\ &= \mathcal{F}^{-1}(\Pi_b) \star (f \cdot \text{Ш}_{1/b}) && \text{iFT of } \text{Ш}_b \end{aligned}$$

which gives the celebrated Shannon-formula

$$f(t) = \text{sinc}(bt) \star \sum_{k \in \mathbb{Z}} f(k/b)\, \delta(t - k/b) = \sum_{k \in \mathbb{Z}} f(k/b)\, \text{sinc}\,(b(t - k/b))$$

where

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} = \mathcal{F}^{-1}(\Pi_1)(t)$$