Orthogonal Filters and Reconstruction
Daubechies and Coiflet filters

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FIR filters

- $\boldsymbol{h} = (h_0, h_1, \ldots, h_L)$: real causal FIR filter of length $L+1$ (where $h_0 \neq 0 \neq h_L$)
- polynomial representation (*z-transform*)

$$h(z) = h_0 + h_1\, z + h_2\, z^2 + \cdots + h_L\, z^L$$

- Fourier series representation (*frequency response*)

$$H(\omega) = h_0 + h_1\, e^{i\omega} + h_2\, e^{2i\omega} \cdots + h_L\, e^{Li\omega} = h(e^{i\omega})$$

Signals

- signal $\boldsymbol{a} = (a[n])_{n\in\mathbb{Z}}$
- power series representation (*z-transform*)

$$a(z) = \sum_{n\in\mathbb{Z}} a[n]\, z^n$$

- Fourier series representation (*frequency representation*)

$$A(\omega) = \sum_{n\in\mathbb{Z}} a[n]\, e^{in\omega} = a(e^{i\omega})$$

Filtering via convolution

- filtering of a signal, $\boldsymbol{a} = (a[n])_{n\in\mathbb{Z}}$ via convolution with $\boldsymbol{h}$

$$\mathcal{T}_{\boldsymbol{h}} : \boldsymbol{a} = (a[n])_{n\in\mathbb{Z}} \longmapsto \boldsymbol{h} \star \boldsymbol{a} = \left(\sum_{k=0}^{L} h_k \, a[n-k] \right)_{n\in\mathbb{Z}}$$

- convolution theorem

$$\mathcal{T}_{\boldsymbol{h}} : a(z) \mapsto h(z) \cdot a(z)$$

- equivalently

$$\mathcal{T}_{\boldsymbol{h}} : A(\omega) \mapsto H(\omega) \cdot A(\omega)$$

Filtering as matrix multiplication

$$\begin{bmatrix} \vdots \\ a[-2] \\ a[-1] \\ a[0] \\ a[1] \\ a[2] \\ \vdots \end{bmatrix} \longmapsto \begin{bmatrix} \ddots & \ddots & \dots & \ddots & \ddots & & & & \\ & h_L & h_{L-1} & \dots & h_1 & h_0 & & & \\ & & h_L & h_{L-1} & \dots & h_1 & h_0 & & \\ & & & h_L & h_{L-1} & \dots & h_1 & h_0 & \\ & & & & \ddots & \ddots & \dots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ a[-2] \\ a[-1] \\ a[0] \\ a[1] \\ a[2] \\ \vdots \end{bmatrix}$$

Filtering by $\boldsymbol{h}$ followed by downsampling $\downarrow_2$

- transformation matrix

$$H = \begin{bmatrix} \ddots & \ddots & \dots & \ddots & \ddots & & & & \\ & h_L & h_{L-1} & h_{L-2} & \dots & h_0 & & & \\ & & & h_L & h_{L-1} & \dots & h_1 & h_0 & \\ & & & & \ddots & \ddots & \dots & \ddots & \ddots \end{bmatrix}$$

Filtering and downsampling

- operating on signals

$$H : \boldsymbol{a} = (a[n])_{n\in\mathbb{Z}} \longmapsto \left(\sum_{k=0}^{L} h_k \, a[2n-k]\right)_{n\in\mathbb{Z}} = \left(\sum_{k=0}^{L} h_{2n-k} \, a[k]\right)_{n\in\mathbb{Z}}$$

- operating on power series

$$H : a(z) \longmapsto \frac{1}{2}\left[\, h(z)\cdot a(z) + h(-z)\cdot a(-z)\,\right]_{z^2\leftarrow z}$$

- operation on Fourier series

$$H : A(\omega) \longmapsto \frac{1}{2}\left[\, H(\frac{\omega}{2})\cdot A(\frac{\omega}{2}) + H(\frac{\omega}{2}+\pi)\cdot A(\frac{\omega}{2}+\pi)\,\right]$$

Adjoint operation $H^\dagger$

- the adjoint operation $H^\dagger$ is realized by the transposed matrix

$$H^{t} = \begin{bmatrix} \ddots & & & \\ \ddots & h_L & & \\ \ddots & h_{L-1} & & \\ & h_{L-2} & h_L & \\ & \vdots & h_{L-1} & \\ & h_0 & \vdots & \\ & & h_1 & \ddots \\ & & h_0 & \ddots \\ & & & \ddots \end{bmatrix}$$

Adjoint operation

- operation on signals

$$H^\dagger : \boldsymbol{a} = (a[n])_{n\in\mathbb{Z}} \longmapsto \left(\sum_{n\le 2k \le L+n} h_{2k-n}\, a[k] \right)_{n\in\mathbb{Z}},$$

- operation on power series

$$H^\dagger : a(z) \longmapsto h(\frac{1}{z}) \cdot a(z^2)$$

- operation on Fourier series

$$H^\dagger : A(\omega) \longmapsto H(-\omega) \cdot A(2\omega) = \overline{H(\omega)} \cdot A(2\omega)$$

- In signal/filter terminology
  - first upsampling $\uparrow_2$,
  - then filtering by $\overleftarrow{\boldsymbol{h}} = (h_{-k})_{k\in\mathbb{Z}}$
- $\overleftarrow{\boldsymbol{h}}$ is not a causal filter: non-zero coefficients in positions $-L, \ldots, 0$

Orthogonality (1)

- $\boldsymbol{h} = (h_0, \ldots, h_L)$ a finite, causal, real filter
- filter length $L+1$ must be even
- H : matrix representing filtering by $\boldsymbol{h}$ followed by downsampling
- the rows of H are *orthogonal*,
  i.e., $H \cdot H^{\mathrm{t}} = I$,
  i.e., the following $\frac{L+1}{2}$ conditions are satisfied

$$(\mathcal{O}_m) \qquad \sum_{k=2m}^{L} h_k \, h_{k-2m} = \delta_{m,0} \qquad (0 \leq m < L/2)$$

Orthogonality (2)

- explanation:

  any two rows of the matrix H have non-zero coefficients in common in $L+1-2m$ positions for $m \in \{0,1,2,\ldots,(L+1)/2\}$

  - For $m = (L+1)/2$ the rows are automatically orthogonal, i.e., for $m \geq (L+1)/2$ condition $(\mathcal{O}_m)$ is satisfied in a trivial way
  - For $1 \leq m < (L+1)/2$ condition $(\mathcal{O}_m)$ expresses orthogonality for rows having $L+1-2m$ non-zero filter positions in common
  - In case $m = 0$ one has condition

    $$h_0^2 + h_1^2 + \cdots + h_L^2 = 1,$$

    i.e., the row vectors of H_N are normalized, i.e., have ℓ^2-Length 1

Overlapping and orthogonality

- Condition ($\mathcal{O}_0$)

h_L	h_{L-1}	h_{L-2}	...	h_1	h_0
h_L	h_{L-1}	h_{L-2}	...	h_1	h_0

- Condition ($\mathcal{O}_1$)

h_L	h_{L-1}	h_{L-2}	h_{L-3}	...	...	h_1	h_0		
		h_L	h_{L-1}	h_{L-2}	...	h_3	h_2	h_1	h_0

- Condition ($\mathcal{O}_2$)

h_L	h_{L-1}	h_{L-2}	h_{L-3}	h_{L-4}	h_{L-5}	...	h_1	h_0				
				h_L	h_{L-1}	...	h_5	h_4	h_3	h_2	h_1	h_0

- Condition ($\mathcal{O}_{\frac{L-1}{2}}$)

h_L	h_{L-1}	...	h_1	h_0			
			h_L	h_{L-1}	...	h_1	h_0

Alternative description of orthogonality

- Filtering of the signal given by $h(1/z)$ using H
- in terms of power series

$$h(z) \cdot h(\frac{1}{z}) + h(-z) \cdot h(-\frac{1}{z}) = 2$$

- in terms of Fourier series

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = H(\omega) \cdot \overline{H(\omega)} + H(\omega + \pi) \cdot \overline{H(\omega + \pi)} = 2$$

Summary

For any filter $\boldsymbol{h} = (h_0, \ldots, h_L)$ the following statements are equivalent:

1. Orthogonality of the rows of H

$$H \cdot H^{t} = I$$

2. Orthogonality conditions

$$(\mathcal{O}_m) \qquad \sum_{k=2m}^{L} h_k \, h_{k-2m} = \delta_{m,0} \qquad (0 \leq m < L/2)$$

3. in terms of power series

$$h(z) \cdot h(1/z) + h(-z) \cdot h(-1/z) = 2$$

4. in terms of Fourier series

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$

Dual filter (1)

- $\boldsymbol{g} = (g_0, \dots, g_L)$ another filter of the same kind with
  $g(z)$ polynomial representation (z-transform)
  $G(\omega)$ Fourier series representation
- Filtering with $\boldsymbol{g}$ followed by downsampling $\downarrow_2$ represented by the matrix

$$G = \begin{bmatrix} \ddots & \ddots & \dots & \ddots & \ddots & & & & & \\ & g_L & g_{L-1} & g_{L-2} & \cdots & g_0 & & & & \\ & & & g_L & g_{L-1} & \cdots & g_1 & g_0 & & \\ & & & & \ddots & \ddots & \dots & \ddots & \ddots \end{bmatrix}.$$

Dual filter (2)

- Operation on signals is given by
  - matrix multiplication

$$G : (a[n])_{n\in\mathbb{Z}} \longmapsto \left(\sum_{k=0}^{L} g_k \, a[2n-k]\right)_{n\in\mathbb{Z}} = \left(\sum_{k=0}^{L} g_{2n-k} \, a[k]\right)_{n\in\mathbb{Z}}$$

  - in terms of power series

$$G : a(z) \longmapsto \frac{1}{2}\big[\, g(z)\cdot a(z) + g(-z)\cdot a(-z)\,\big]_{z^2 \leftarrow z}$$

  - in terms of Fourier series

$$G : A(\omega) \longmapsto \frac{1}{2}\big[\, G(\frac{\omega}{2})\cdot A(\frac{\omega}{2}) + G(\frac{\omega}{2}+\pi)\cdot A(\frac{\omega}{2}+\pi)\,\big]$$

Dual filter (3)

Equivalent statements:

- Orthogonality of the rows of G

$$G \cdot G^{t} = I$$

- Orthogonality conditions

$$(\mathcal{O}'_m) \qquad \sum_{k=2m}^{L} g_k \, g_{k-2m} = \delta_{m,0} \qquad (0 \leq m < L/2)$$

- in terms of power series

$$g(z) \cdot g(1/z) + g(-z) \cdot g(-1/z) = 2$$

- in terms of Fourier series

$$\left|G(\omega)\right|^2 + \left|G(\omega + \pi)\right|^2 = 2$$

Orthogonality of G and H

- Orthogonality of the rows of the matrices

$$H \cdot G^{\mathtt{t}} = 0 \quad \text{equivalently} \quad G \cdot H^{\mathtt{t}} = 0$$

- Orthogonality conditions

$$(\mathcal{O}''_m) \qquad \sum_{k=2m}^{L} h_k \, g_{k-2m} = 0 \qquad (0 \leq m < L/2)$$

- in terms of power series

$$h(z) \cdot g(1/z) + h(-z) \cdot g(-1/z) = 0$$

- in terms of Fourier series

$$H(\omega) \cdot \overline{G(\omega)} + H(\omega + \pi) \cdot \overline{G(\omega + \pi)} = 0$$

Reconstruction (1)

- $\boldsymbol{h}$ and $\boldsymbol{g}$ filters as described
- H and G, and $H^\dagger$ and $G^\dagger$ corresponding transformation matrices
- Reconstruction condition for the filter pair $(\boldsymbol{g}, \boldsymbol{h})$, is

$$H^{\mathrm{t}} \cdot H + G^{\mathrm{t}} \cdot G = I,$$

- in terms of power series

$$h(z) \cdot h(\frac{1}{z}) + g(z) \cdot g(\frac{1}{z}) = 2$$
$$h(z) \cdot h(-\frac{1}{z}) + g(z) \cdot g(-\frac{1}{z}) = 0$$

- in terms of Fourier series

$$|H(\omega)|^2 + |G(\omega)|^2 = 2$$
$$H(\omega) \cdot \overline{H(\omega + \pi)} + G(\omega) \cdot \overline{G(\omega + \pi)} = 0$$

Reconstruction (2)

Justification

- Composition $H^\dagger \circ H$ gives

$$H^\dagger \circ H : a(z) \longmapsto \frac{1}{2} \left(a(z) \cdot h(z) + a(-z) \cdot h(-z) \right) \cdot h(\frac{1}{z})$$

- Composition $G^\dagger \circ G$ gives

$$G^\dagger \circ G : a(z) \longmapsto \frac{1}{2} \left(a(z) \cdot g(z) + a(-z) \cdot g(-z) \right) \cdot g(\frac{1}{z})$$

- Putting these together gives

$$\begin{aligned} H^\dagger \circ H + G^\dagger \circ G : a(z) \longmapsto \ & \frac{1}{2} \left(h(z) \cdot h(\frac{1}{z}) + g(z) \cdot g(\frac{1}{z}) \right) \cdot a(z) \\ & + \frac{1}{2} \left(h(-z) \cdot h(\frac{1}{z}) + g(-z) \cdot g(\frac{1}{z}) \right) \cdot a(-z) \end{aligned}$$

- The coefficient of $a(z)$ must be 1, the coefficient of $a(-z)$ must vanish

Reconstruction (3)

- Looking at filter coefficients,reconstruction is expressed by

$$\sum_{k\in\mathbb{Z}} h_{m-2k} \cdot h_{n-2k} + \sum_{k\in\mathbb{Z}} g_{m-2k} \cdot g_{n-2k} = \delta_{m,n}$$

Reconstruction (4)

- Theorem:
  - $\boldsymbol{h} = (h_0, \dots, h_L)$ orthogonal filter of even length $L + 1$, i.e., $H \cdot H^{\mathrm{t}} = I$
  - $\boldsymbol{g} = (g_0, \dots, g_L)$ dual filter defined by

$$g_k = (-1)^k h_{L-k} \qquad (0 \le k \le L).$$

  Then the following holds:

  1. $\boldsymbol{g}$ is an orthogonal filter, i.e.,

$$G \cdot G^{\mathrm{t}} = I$$

  2. filters $\boldsymbol{g}$ and $\boldsymbol{h}$ are orthogonal, i.e.,

$$H \cdot G^{\mathrm{t}} = 0 = G \cdot H^{\mathrm{t}}$$

  3. the condition for reconstruction is satisfied, i.e.,

$$H^{\mathrm{t}} \cdot H + G^{\mathrm{t}} \cdot G = I$$

Reconstruction (5)

About the *proof*:

- The definition of the filter $\boldsymbol{g}$ can be written as

$$
\begin{aligned}
g(z) &= \sum_{k=0}^{L} g_k \, z^k = \sum_{k=0}^{L} h_{k-L} \, (-z)^k \\
&= \sum_{k=0}^{L} h_k \, (-z)^{L-k} = (-z)^L \sum_{k=0}^{L} h_k \, (-\frac{1}{z})^k \\
&= (-z)^L \, h(-\frac{1}{z})
\end{aligned}
$$

- Orthogonality of $\boldsymbol{g}$:

$$
\begin{aligned}
g(z)\cdot g(\frac{1}{z}) + g(-z)\cdot g(-\frac{1}{z}) \\
&= (-z)^L\, h(-\frac{1}{z})\cdot(-\frac{1}{z})^L\, h(-z) + z^L\, h(\frac{1}{z})\cdot(\frac{1}{z})^L\, h(z) \\
&= h(-\frac{1}{z})\cdot h(-z) + h(\frac{1}{z})\cdot h(z) = 2
\end{aligned}
$$

- Orthogonality of $\boldsymbol{g}$ und $\boldsymbol{h}$:

$$
\begin{aligned}
h(z)\cdot g(\frac{1}{z}) + h(-z)\cdot g(-\frac{1}{z}) \\
&= h(z)\cdot(-\frac{1}{z})^L\, h(-z) + h(-z)\cdot(\frac{1}{z})^L\, h(z) \\
&= (\frac{1}{z})^L\left[(-1)^L\, h(z)\cdot h(-z) + h(-z)\cdot h(z)\right] = 0,
\end{aligned}
$$

(since L is odd)

Reconstruction (6)

- Reconstruction condition:

$$
\begin{aligned}
h(z) \cdot h(\frac{1}{z}) + g(z) \cdot g(\frac{1}{z}) \\
&= h(z) \cdot h(\frac{1}{z}) + (-z)^L\, h(-\frac{1}{z}) \cdot (-\frac{1}{z})^L h(-z) \\
&= h(z) \cdot h(\frac{1}{z}) + h(-\frac{1}{z}) \cdot h(-z) = 2
\end{aligned}
$$

$$
\begin{aligned}
h(z) \cdot h(-\frac{1}{z}) + g(z) \cdot g(-\frac{1}{z}) \\
&= h(z) \cdot h(-\frac{1}{z}) + (-z)^L\, h(-\frac{1}{z}) \cdot (\frac{1}{z})^L h(z) \\
&= h(z) \cdot h(-\frac{1}{z}) + (-1)^L\, h(-\frac{1}{z}) \cdot h(z) = 0,
\end{aligned}
$$

(since L is odd)

Reconstruction (5)

- From the reconstruction condition on gets the *filter bank* setup:
  - *Analysis*

    a signal $\boldsymbol{a}$ is decomposed via filtering with $\boldsymbol{h}$ and $\boldsymbol{g}$ into two signals $\boldsymbol{b} = H \cdot \boldsymbol{a}$ and $\boldsymbol{c} = G \cdot \boldsymbol{a}$
  - *Synthesis*

    from these signals $\boldsymbol{b}$ and $\boldsymbol{c}$ the signal $\boldsymbol{a}$ can be reconstructed

$$\boldsymbol{a} \longmapsto (\boldsymbol{b}, \boldsymbol{c}) = (H \cdot \boldsymbol{a}, G \cdot \boldsymbol{a}) \longmapsto \begin{cases} H^{\mathtt{t}} \cdot \boldsymbol{b} + G^{\mathtt{t}} \cdot \boldsymbol{c} = \\ (H^{\mathtt{t}} \cdot H + G^{\mathtt{t}} \cdot G) \cdot \boldsymbol{a} \quad = \boldsymbol{a} \end{cases}$$

Finite-length signals (1)

Filtering with $\boldsymbol{h} = (h_0, \ldots, l_L)$ followed by downsampling $\downarrow_2$ of signals of finite (even) length N is given by matrix multiplication with an $(N/2) \times N$ matrix H_N of cyclic structure
("overshooting" rows will be cyclically wrapped)

$$H_N =$$

$$\begin{bmatrix}
h_L & h_{L-1} & h_{L-2} & \ldots & \ldots & h_1 & h_0 & & & & & \\
 & & h_L & h_{L-1} & \ldots & \ldots & \ldots & h_1 & h_0 & & & \\
 & & & \ddots & \ddots & & & & \ddots & \ddots & & \\
 & & & & h_L & h_{L-1} & \ldots & \ldots & \ldots & \ldots & h_1 & h_0 \\
h_1 & h_0 & & & & & h_L & h_{L-1} & \ldots & \ldots & h_3 & h_2 \\
h_3 & h_2 & h_1 & h_0 & & & & & h_L & \ldots & h_5 & h_4 \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & & & \ddots & \vdots & \vdots \\
h_{L-2} & h_{L-3} & \ldots & \ldots & h_1 & h_0 & & & & & h_L & h_{L-1}
\end{bmatrix}$$

Finite-length signals (2)

- *cyclic wrapping* allows to relate properties of the infinite matrix H to properties of H_N:
  - If H is an orthogonal matrix, then so is H_N (the converse holds provided $N \geq 2L$)
  - All previously introduced ways of expressing orthogonality and reconstruction in terms of polynomials, power series amd Fourier series are thus available

Finite-length signals (3)

- Multiplication with matrices H_N, H_N^t (similarly G_N and G_N^t) written out explicitly:
  - Multiplication of a column vector $\boldsymbol{v} = (v_k)_{0 \le k < N}$ with matrix H_N (from the left):

$$H_N \cdot \boldsymbol{v} = \boldsymbol{w} = (w_j)_{0 \le j < N/2} \quad \text{where} \quad w_j = \sum_{k=0}^{L} h_k \, v_{2j+L-k \bmod N}.$$

  - adjoint transformation: multiplication of a column vector $\boldsymbol{w} = (w_j)_{0 \le j < N/2}$ with the transposed matrix H_N^t (from the left):

$$H_N^t \cdot \boldsymbol{w} = \boldsymbol{v} = (v_j)_{0 \le j < N} \quad \text{where} \quad \begin{cases} v_{2j} = \displaystyle\sum_{k=0}^{(L-1)/2} h_{2k+1} \, w_{k+j-\frac{L-1}{2} \bmod N} \\ v_{2j+1} = \displaystyle\sum_{k=0}^{(L-1)/2} h_{2k} \, w_{k+j-\frac{L-1}{2} \bmod N} \end{cases}$$

Finite-length signals (4)

- *Analysis*
  If $\boldsymbol{h}$, $\boldsymbol{g}$ are orthogonal filter of the same (even) length $L+1$ which are orthogonal to each other, then for vectors of even length $N \geq L+1$ one has the orthogonal transform

$$\boldsymbol{a} \longmapsto \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} H_N \cdot \boldsymbol{a} \\ G_N \cdot \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} H_N \\ G_N \end{bmatrix} \cdot \boldsymbol{a}.$$

- *Synthesis*
  If the reconstruction condition is satisfied, one can recover $\boldsymbol{a}$ from $\boldsymbol{b}$ und $\boldsymbol{c}$

$$\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} \longmapsto \begin{bmatrix} H_N \\ G_N \end{bmatrix}^{\mathtt{t}} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} H_N^{\mathtt{t}} & G_N^{\mathtt{t}} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix}$$

$$= H_N^{\mathtt{t}} \cdot \boldsymbol{b} + G_N^{\mathtt{t}} \cdot \boldsymbol{c} = \left(H_N^{\mathtt{t}} \cdot H_N + G_N^{\mathtt{t}} \cdot G_N \right) \cdot \boldsymbol{a} = \boldsymbol{a}.$$

Daubechies filters

- In 1988 Ingrid DAUBECHIES[1] invented a procedure for constructing orthogonal filter pairs $(\boldsymbol{h}, \boldsymbol{g})$ of the same (even) length having high- and low-pass properties
- These filters enjoy interesting (and desirable!):
  - the scaling and wavelet functions ϕ und ψ associated to them have compact support, i.e., they vanish outside a finite interval
  - by increasing the filter length one obtains increasingly smooth (higher differentiability) wavelet functions

[1] I. DAUBECHIES, Orthonormal bases for compactly supported wavelets, *Comm. Pure. Appl. Math.* 41:909–996, 1988. Orthonormal bases for compactly supported wavlets II, *SIAM J. Math. Anal.* 24(23):499–519. *Ten Lectures on Wavelets*, SIAM, 1992.

Construction of the D_4 filter pair (1)

- Goal: constructing a filter pair $(\boldsymbol{h}, \boldsymbol{g})$ of filters of length 4 s.th.
  - $\boldsymbol{h} = (h_0, h_1, h_2, h_3)$ acts as a low-pass filter
  - $\boldsymbol{g} = (g_0, g_1, g_2, g_3)$ acts as a high-pass filter
- The corresponding Fourier series are

$$H(\omega) = h_0 + h_1\, e^{i\omega} + h_2\, e^{2i\omega} + h_3\, e^{3i\omega}$$
$$G(\omega) = g_0 + g_1\, e^{i\omega} + g_2\, e^{2i\omega} + g_3\, e^{3i\omega}$$

Construction of the D_4 filter pair (2)

- The transformation matrices for signals of length N are

$$H_N = \begin{bmatrix} h_3 & h_2 & h_1 & h_0 & & & & & & \\ & & h_3 & h_2 & h_1 & h_0 & & & & \\ & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & h_3 & h_2 & h_1 & h_0 \\ h_1 & h_0 & & & & & & & h_3 & h_2 \end{bmatrix}$$

$$G_N = \begin{bmatrix} g_3 & g_2 & g_1 & g_0 & & & & & & \\ & & g_3 & g_2 & g_1 & g_0 & & & & \\ & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & g_3 & g_2 & g_1 & g_0 \\ g_1 & g_0 & & & & & & & g_3 & g_2 \end{bmatrix}$$

Construction of the D_4 filter pair (3)

- W_N: the transformation matrix for signals of length N of the corresponding wavelet transform contains the low-pass filter $\boldsymbol{h}$ and the high-pass filter $\boldsymbol{g}$:

$$W_N = \begin{bmatrix} H_N \\ G_N \end{bmatrix}$$

- The adjoint (= transposed) matrix of W_N is

$$W_N^{\mathrm{t}} = \begin{bmatrix} H_N^{\mathrm{t}} & G_N^{\mathrm{t}} \end{bmatrix}$$

Construction of the D_4 filter pair (4)

- The first important condition is
  - The transformation matrix W_N shall be orthogonal, i.e.,

$$W_N \cdot W_N^{\mathtt{t}} = I_N$$

  - Written out:

$$\begin{aligned} W_N \cdot W_N^{\mathtt{t}} &= \begin{bmatrix} H_N \\ G_N \end{bmatrix} \cdot \begin{bmatrix} H_N^{\mathtt{t}} & G_N^{\mathtt{t}} \end{bmatrix} \\ &= \begin{bmatrix} H_N \cdot H_N^{\mathtt{t}} & H_N \cdot G_N^{\mathtt{t}} \\ G_N \cdot H_N^{\mathtt{t}} & G_N \cdot G_N^{\mathtt{t}} \end{bmatrix} = \begin{bmatrix} I_{N/2} & 0_{N/2} \\ 0_{N/2} & I_{N/2} \end{bmatrix} \end{aligned}$$

Construction of the D_4 filter pair (5)

- There are three types of orthogonality conditions to be satisfied:
  1. $H_N \cdot H_N^{t} = I_{N/2}$ (orthogonality of the rows of H_N)

$$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\ h_0 h_2 + h_1 h_3 &= 0 \end{aligned}$$

  2. $G_N \cdot G_N^{t} = I_{N/2}$ (orthogonality of the rows of G_N)

$$\begin{aligned} g_0^2 + g_1^2 + g_2^2 + g_3^2 &= 1 \\ g_0 g_2 + g_1 g_3 &= 0 \end{aligned}$$

  3. $H_N \cdot G_N^{t} = 0_{N/2}$ (orthogonality of the rows of G_N and H_N)

$$\begin{aligned} h_0 g_0 + h_1 g_1 + h_2 g_2 + h_3 g_3 &= 0 \\ h_0 g_2 + h_1 g_3 &= 0 \\ h_2 g_0 + h_3 g_1 &= 0 \end{aligned}$$

Construction of the D_4 filter pair (6)

- Type 3 is easily satisfied if one puts

$$\boxed{g_j = (-1)^j h_{3-j} \ (0 \le j \le 3)}$$

- With this choice conditions 1. and 2. become equivalent, so that it remains to satisfy condition 1
- Specifying the low-pass condition for $\boldsymbol{h}$ and the high-pass condition for $\boldsymbol{g}$ is done by:
  - $\boldsymbol{h}$ is a low-pass filter: $\boxed{H(\pi) = 0}$
  - $\boldsymbol{g}$ is a high-pass filter : $\boxed{G(0) = 0}$
- Both conditions are equivalent in view of the imposed relation between g_j and h_j:

$$H(\pi) = h_0 - h_1 + h_2 - h_3 = 0 \iff$$

$$G(0) = g_0 + g_1 + g_2 + g_3 = h_3 - h_2 + h_1 - h_0 = 0$$

Construction of the D_4 filter pair (7)

- It remains to determine $\boldsymbol{h} = (h_0, h_1, h_2, h_3)$ such that

$$
\begin{aligned}
(\mathcal{O}_0) \qquad & h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\
(\mathcal{O}_1) \qquad & h_0 h_2 + h_1 h_3 = 0 \\
(\mathcal{T}_0) \qquad & h_0 - h_1 + h_2 - h_3 = 0
\end{aligned}
$$

- A consequence of these three conditions is

$$H(0) = G(\pi) = h_0 + h_1 + h_2 + h_3 = \pm\sqrt{2}$$

- So one is left with three conditions for the four coefficients h_0, h_1, h_2, h_3 to be determined.

  One expects a one-parameter solution set

Construction of the D_4 filter pair (8)

- It follows from $(\mathcal{O}_1)$ that

$$(h_2, h_3) = c \cdot (-h_1, h_0)$$

for some $c \in \mathbb{R}, c \neq 0$

- From $(\mathcal{O}_0)$

$$h_0^2 + h_1^2 = \frac{1}{1+c^2}, \quad \text{and thus} \quad h_1 = \frac{1-c}{1+c} \cdot h_0$$

- Furthermore

$$h_0^2 = \frac{(1+c)^2}{2(1+c^2)^2}$$

- From the two possibilities given by

$$h_0 = \pm \frac{1+c}{\sqrt{2}(1+c^2)}$$

one choses the one with the positive sign

Construction of the D_4 filter pair (9)

- Thus one arrives at the solution

$$h_0 = \frac{1+c}{\sqrt{2}(1+c^2)}$$
$$h_1 = \frac{1-c}{\sqrt{2}(1+c^2)}$$
$$h_2 = \frac{-c(1-c)}{\sqrt{2}(1+c^2)}$$
$$h_3 = \frac{c(1+c)}{\sqrt{2}(1+c^2)}.$$

Construction of the D_4 filter pair (10)

- In order to fix the value of the parameter c a second low-pass condition is introduced:

$$H'(\pi) = 0$$

- For the filter coefficients this means

$$(\mathcal{T}_1) \quad h_1 - 2h_2 + 3h_3 = 0$$

- which can be written as

$$(1 + 2c)h_1 + 3c_0 = 0$$

and from

$$h_1 = \frac{1 - c}{1 + c} \cdot h_0$$

this finally leads to

$$\frac{1 - c}{1 + c} = -\frac{3c}{1 + 2c}$$

Construction of the D_4 filter pair (11)

- One gets

$$c^2 + 4c + 1 = 0$$

- from which one takes the solution $c = -2 + \sqrt{3}$, so that

$$h_0 = \pm \frac{1+\sqrt{3}}{4\sqrt{2}}$$

- Taking the positive sign one finally obtains

$$h_0 = \frac{1}{4\sqrt{2}}(1+\sqrt{3}) \qquad g_0 = \frac{1}{4\sqrt{2}}(1-\sqrt{3})$$
$$h_1 = \frac{1}{4\sqrt{2}}(3+\sqrt{3}) \qquad g_1 = \frac{-1}{4\sqrt{2}}(3-\sqrt{3})$$
$$h_2 = \frac{1}{4\sqrt{2}}(3-\sqrt{3}) \qquad g_2 = \frac{1}{4\sqrt{2}}(3+\sqrt{3})$$
$$h_3 = \frac{1}{4\sqrt{2}}(1-\sqrt{3}) \qquad g_3 = \frac{-1}{4\sqrt{2}}(1+\sqrt{3})$$

This is the D_4 filter pair

Construction of the D_6 filter pair (1)

- The construction of a filter pair $(\boldsymbol{h}, \boldsymbol{g})$ with $\boldsymbol{h} = (h_0, h_1, \ldots, h_5)$ and $\boldsymbol{g} = (g_0, g_1, \ldots, g_5)$ proceeds along the same lines
- The filters to be determined are required to be related by

$$\boxed{g_j = (-1)^j h_{5-j} \quad (0 \le j \le 5)}$$

$\Rightarrow$ many orthogonality conditions are automatically satisfied

- Three orthogonality conditions remain to be satisfied:

$$\begin{array}{ll} (\mathcal{O}_0) & h_0^2 + h_1^2 + h_2^2 + h_3^2 + h_4^2 + h_5^2 = 1 \\ (\mathcal{O}_1) & h_0 h_2 + h_1 h_3 + h_2 h_4 + h_3 h_5 = 0 \\ (\mathcal{O}_2) & h_0 h_4 + h_1 h_5 = 0 \end{array}$$

- The low-pass properties of $\boldsymbol{h}$ are specified as follows:

$$\begin{array}{llll} (\mathcal{T}_0) & H(\pi) = 0 & \Leftrightarrow & h_0 - h_1 + h_2 - h_3 + h_4 - h_5 = 0 \\ (\mathcal{T}_1) & H'(\pi) = 0 & \Leftrightarrow & h_1 + 2h_2 - 3h_3 + 4h_4 - 5h_5 = 0 \\ (\mathcal{T}_2) & H''(\pi) = 0 & \Leftrightarrow & h_1 + 4h_2 - 9h_3 + 16h_4 - 25h_5 = 0 \end{array}$$

Construction of the D_6 filter pair (2)

- A real solution of these 6 conditions $(\mathcal{O}_0), (\mathcal{O}_1), (\mathcal{O}_2), (\mathcal{T}_0), (\mathcal{T}_1), (\mathcal{T}_2)$ for $h_0, \ldots, h_5$ is given by

$$h_0 = \frac{\sqrt{2}}{32}\left(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}\right) \approx 0.332671$$

$$h_1 = \frac{\sqrt{2}}{32}\left(5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}}\right) \approx 0.806892$$

$$h_2 = \frac{\sqrt{2}}{32}\left(10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}}\right) \approx 0.459878$$

$$h_3 = \frac{\sqrt{2}}{32}\left(10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}}\right) \approx -0.135011$$

$$h_4 = \frac{\sqrt{2}}{32}\left(5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}}\right) \approx -0.085441$$

$$h_5 = \frac{\sqrt{2}}{32}\left(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}\right) \approx 0.035226$$

These are the coefficients of the low-pass filter of the D_6 filter pair

Construction of the D_{2M} filter pair (1)

- Now let $L = 2M - 1$. The construction should yield filter pairs $(\boldsymbol{h}, \boldsymbol{g})$ with $\boldsymbol{h} = (h_0, h_1, \ldots, h_L)$, $\quad \boldsymbol{g} = (g_0, g_1, \ldots, g_L)$, where

$$\boxed{g_j = (-1)^j \, h_{L-j} \quad (0 \leq j \leq L)}$$

- The relevant M *orthogonality conditions* are:

$$\boxed{(\mathcal{O}_m) \qquad \sum_{k=2m}^{L} h_k \, h_{k-2m} = \delta_{m,0} \quad (0 \leq m < M)}$$

- For the Fourier series $H(\omega) = \sum_{k=0}^{L} h_k \, e^{ik\omega}$ this amounts to

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$

Construction of the D_{2M} filter pair (2)

- Furthermore, there are M *low-pass conditions*, which are specified using the derivatives of the Fourier series $H(\omega)$ at $\omega = \pi$:

$$(\mathcal{T}_m) \qquad H^{(m)}(\pi) = 0 \quad (0 \le m < M).$$

- For the filter coefficients these are the *moment conditions*

$$\boxed{(T_m) \qquad \sum_{k=0}^{L} (-1)^k \, k^m \, h_k = 0 \quad (0 \le m < M)}$$

- In total one has $2M = L + 1$ conditions for the $L + 1$ coefficients $h_0, h_1, \ldots, h_L$, of which
  - M are linear (*low-pass*) and
  - M are non-linear (quadratic, *orthogonality*)
- One always has

$$H(0) = \sum_{k=0}^{L} h_k = \pm\sqrt{2}$$

Construction of the D_{2M} filter pair (3)

- The low-pass conditions can be viewed algebraically by considering the polynomial ("z-transform")

$$h(z) = \sum_{k=0}^{L} h_k \, z^k, \quad \text{so that} \quad H(\omega) = h(e^{i\omega})$$

- The low-pass conditions are then equivalent to
  - For $z = -1$ the polynomial $h(z)$ has a root of multiplicity $> M$
- Another equivalent statement is
  - $h(z) = (z+1)^M \cdot q(z)$ for some polynomial $q(z)$ of degree $M - 1$

Construction of the D_{2M} filter pair (4)

Theorem (Daubechies)

- The system consisting of
  - the M orthogonality conditions $(\mathcal{O}_m)_{0 \leq m < M}$ and the
  - the M low-pass conditions $(\mathcal{T}_M)_{0 \leq m < M}$

  for filters of length $2M$ has $2^{\lfloor (2M+1)/4 \rfloor}$ real solutions
- There is exactly one (!) solution for which $|z_k| > 1$ holds for all roots of the corresponding polynomial $q(z)$
- This solution specifies the Daubechies low-pass filter $\boldsymbol{h}$ von D_{2M}

Construction of the D_{2M} filter pair (5)

- The Daubechies low-pass filter D_2 with $\boldsymbol{h} = (h_0, h_1)$ is determined via the conditions

$$h_0^2 + h_1^2 = 1, \quad h_0 - h_1 = 0$$

- Consequently

$$\boldsymbol{h} = (1/\sqrt{2}, 1/\sqrt{2}), \quad \boldsymbol{g} = (1/\sqrt{2}, -1/\sqrt{2}).$$

- This is nothing but the HAAR-filter pair!

Construction of the D_{2M} filter pair (6)

- Actually, constructing DAUBECHIES filters is not a simple task!
- Let $L = 2M - 1$. One wants a filter $\boldsymbol{h} = (h_0, h_1, \ldots, h_L)$ for which the orthogonality condition

$$h(z) \cdot h(\frac{1}{z}) + h(-z) \cdot h(-\frac{1}{z}) = 2$$

  is satisfied
- On the complex unit circle one has $\overline{z} = 1/z$. Since the filter coefficients should be *real*, one may write

$$|h(z)|^2 + |h(-z)|^2 = 2 \quad \text{for} \quad |z| = 1$$

- The low-pass conditions require that

$$h(z) = (1 + z)^M \cdot q(z)$$

  for some real polynomial $q(z)$ of degree $M - 1$

Construction of the D_{2M} filter pair (7)

Some cosmetic modifications:

- Instead of $h(z)$ consider the polynomial $h(z)/\sqrt{2}$, so that the “2” on the right-hand side of the orthogonality condition can be replaced by a “1”
- The modified polynomial shall be written as

$$\widetilde{h}(z) = \frac{1}{\sqrt{2}}\, h(z) = \left(\frac{1+z}{2}\right)^M \cdot q_{M-1}(z)$$

- which does not change the roots of the involved polynomials
- The subscript $M-1$ of the polynomial on the right indicates its degree, which will be practiced in what follows

Construction of the D_{2M} filter pair (8)

- For z on the complex unit circle, i.e., $z = e^{i\phi}$ one has

$$\left|\frac{1+z}{2}\right|^2 = \frac{1+\cos\phi}{2} = 1 - \sin^2\frac{\phi}{2},$$
$$\left|\frac{1-z}{2}\right|^2 = \frac{1-\cos\phi}{2} = \sin^2\frac{\phi}{2}.$$

- The equation

$$\left|\widetilde{h}(z)\right|^2 + \left|\widetilde{h}(-z)\right|^2 = 1$$

for $|z| = 1$ can be written as

$$\left(1-\sin^2\frac{\phi}{2}\right)^M \cdot \left|q_{M-1}(e^{i\phi})\right|^2 + \left(\sin^2\frac{\phi}{2}\right)^M \cdot \left|q_{M-1}(-e^{i\phi})\right|^2 = 1.$$

Construction of the D_{2M} filter pair (9)

- Since $q_{M-1}(z)$ should be a polynomial with *real* coefficients,
  - $\left|q_{M-1}(e^{i\phi})\right|^2$ can be written as a polynomial in $\cos\phi$,
  - and also as a polynomial (of degree $M-1$) in $1-\sin^2\frac{\phi}{2}$,
  - and as a polynomial $p_{M-1}(y)$ in $y=\sin^2\frac{\phi}{2}$
- Between the new variable $y=\sin^2\frac{\phi}{2}$ and the original variable $z=e^{i\phi}$ one has the relation

$$y=\frac{1}{2}-\frac{1}{4}\left(z+\frac{1}{z}\right)$$

Construction of the D_{2M} filter pair (10)

- From
$$\left|q_{M-1}(e^{i\phi})\right|^2 = p_{M-1}(y)$$
- one has
$$\left|q_{M-1}(-e^{i\phi})\right|^2 = p_{M-1}(1-y).$$
- To summarize: one is looking for a polynomial $p_{M-1}(y)$ with the two properties:
  - $(1-y)^M \cdot p_{M-1}(y) + y^M \cdot p_{M-1}(1-y) = 1,$
  - $p_{M-1}(y) \geq 0$ for $0 \leq y \leq 1$

Construction of the D_{2M} filter pair (11)

- The *Daubechies polynomials* $P_M(y)$ are defined as

$$P_M(y) = \sum_{m=0}^{M} \binom{M+m}{m} y^m.$$

- The first few of these polynomials are

$$
\begin{aligned}
P_0(y) &= 1 \\
P_1(y) &= 1 + 2\,y \\
P_2(y) &= 1 + 3\,y + 6\,y^2 \\
P_3(y) &= 1 + 4y + 10y^2 + 20y^3
\end{aligned}
$$

Construction of the D_{2M} filter pair (12)

- These polynomials can be written as

$$P_M(y) = \sum_{k=0}^{M} \binom{2M+1}{k} y^k \, (1-y)^{M-k}$$

(see Lecture Notes)

- Claim: The Daubechies polynomials satisfy

$$(1-y)^{M+1} \cdot P_M(y) + y^{M+1} \cdot P_M(1-y) = 1$$

- Obviously $P_M(y) \geq 0$ for $0 \leq y \leq 1$

Construction of the D_{2M} filter pair (13)

Proof of the claim:

- Use the binomial formula to obtain

$$
\begin{aligned}
&(1-y)^{M+1} \cdot P_M(y) + y^{M+1} \cdot P_M(1-y) \\
&= \sum_{k=0}^{M} \binom{2M+1}{k} y^k (1-y)^{2M+1-k} + \sum_{k=0}^{M} \binom{2M+1}{k} (1-y)^k y^{2M+1-k} \\
&= \sum_{k=0}^{2M-1} \binom{2M+1}{k} y^k (1-y)^{2M+1-k} = \big(y + (1-y)\big)^{2M+1} = 1
\end{aligned}
$$

Construction of the D_{2M} filter pair (14)

- Now let

$$\widehat{P}_{2M-1}(z) = (1-y)^M \cdot P_{M-1}(y) = \sum_{m=-2M+1}^{2M-1} a_m \, z^m,$$

- The relation between y and z is

$$y = \frac{1}{2} - \frac{1}{4}\left(z + \frac{1}{z}\right)$$

- For $\widehat{P}_{2M-1}(z)$ one has
  1. For $z \in \mathbb{C}_{\neq 0}$:
  $$\widehat{P}_{2M-1}(z) + \widehat{P}_{2M-1}(-z) = 1$$
  2. For $z \in \mathbb{C}_{\neq 0}$:
  $$\widehat{P}_{2M-1}(z) = \widehat{P}_{2M-1}(1/z)$$
  3. For $z \in \mathbb{C}$ with $|z| = 1$: $\widehat{P}_{2M-1}(z) \geq 0$

Construction of the D_{2M} filter pair (15)

- $\widehat{P}_{2M-1}(z)$ is a “Laurent polynomial”, in which monomials with negative exponents may appear
- This can be turned into a polynomial by putting

$$\mathbf{P}_{4M-2}(z) = z^{2M-1} \cdot \widehat{P}_{2M-1}(z)$$

- $\mathbf{P}_{4M-2}(z)$ has $z = -1$ as root of multiplicity $2M$ and one has $\mathbf{P}_{4M-2}(1) = 1$
- If $z_0 \in \mathbb{C}_{\neq 0}$ is a root of $\mathbb{P}_{4M-2}(z)$, then so are $\overline{z_0}$, $1/z_0$ and $1/\overline{z_0}$, and they are of the same order

Construction of the D_{2M} filter pair (16)

- If $z_0 \neq 0$ *real*, then for $|z| = 1$:

$$\left|(z - z_0)(z - z_0^{-1})\right| = \frac{1}{|z_0|} \left|z - z_0\right|^2 .$$

- If $z_0 \neq 0$ is *not real*, then for $|z| = 1$:

$$\left|(z - z_0)(z - \overline{z_0}^{-1})(z - \overline{z_0})(z - z_0^{-1})\right| = \frac{1}{|z_0|^2} \left|z - z_0\right|^2 \left|z - \overline{z_0}\right|^2$$

Construction of the D_{2M} filter pair (17)

This leads to the desired result:

- There exists a real polynomial $\mathbf{Q}_{M-1}(z)$ s.th.

$$\mathbf{P}_{2M-1}(z) = \left|\frac{1+z}{2}\right|^{2M} \cdot \left|\mathbf{Q}_{M-1}(z)\right|^2$$

- This needs to be shown only for $|z| = 1$, it then follows for all complex z
- For $|z| = 1$ the assertion follows from the previous theorem by grouping together corresponding roots

Once again D_4

- We have
$$P_2(y) = 1 + 3y + 6y^2$$
- Substitution gives
$$\mathbf{P}_6(z) = -\frac{1}{32}\left(-z^6 + 9\,z^4 + 16\,z^3 + 9\,z^2 - 1\right).$$
- This can be factored into
$$-\frac{1}{32}\,\left(z^2 - 4\,z + 1\right)(z+1)^4$$
and this exhibits $z = -1$ as a root of multiplicity 4
- The quadratic factor has (real) roots $z = 2 \pm \sqrt{3}$
- Setting $\alpha = 2 - \sqrt{3}$ one obtains

$$h(z) = \frac{1}{4}\,\frac{(z+1)^2\left(z - 2 + \sqrt{3}\right)}{1/2\,\sqrt{6} - 1/2\,\sqrt{2}}$$
$$\approx 0.48296291\,z^3 + 0.83651630\,z^2 + 0.2241438\,z - 0.12940952$$

Once again D_6 (1)

- We have

$$P_3(y) = 1 + 4y + 10y^2 + 20y^3$$

- Substitution gives

$$\mathbf{P}_{10}(z) = \frac{1}{512}\left(3\,z^{10} - 25\,z^8 + 150\,z^6 + 256\,z^5 + 150\,z^4 - 25\,z^2 + 3\right)$$

- This can be factored into

$$\frac{1}{512}\left(3\,z^4 - 18\,z^3 + 38\,z^2 - 18\,z + 3\right)(z+1)^6$$

and this exhibits $z = -1$ as a root of multiplicity 6

Once again D_6 (2)

- The factor of degree 4 has roots

$$\begin{aligned}
\alpha &= 0.2872513780 + 0.1528923339\, i, \\
\alpha^{-1} &= 2.712748622 - 1.443886783\, i, \\
\overline{\alpha} &= 0.2872513780 - 0.1528923339\, i, \\
\overline{\alpha}^{-1} &= 2.712748622 + 1.443886783\, i
\end{aligned}$$

- This gives

$$\begin{aligned}
h(z) &= \frac{\sqrt{3}}{16\,|\alpha|} \cdot (z+1)^3 \cdot (z-\alpha) \cdot (z-\overline{\alpha}) \\
&\approx 0.3326705530\, z^5 + 0.8068915095\, z^4 + 0.4598775023\, z^3 \\
&\quad - 0.1350110200\, z^2 - 0.08544127389\, z + 0.03522629187
\end{aligned}$$

Non-causal filters (1)

- Consider more generally finite filters $\boldsymbol{h} = (h_\ell, h_{\ell+1}, \ldots, h_L)$ with $\ell < L$ and $\ell \leq 0 \leq L$, so that the filter has length $L - \ell + 1$
- Because of 2-downsampling the filter length must be even, $L - \ell + 1 = 2M$ say, so that $\ell \not\equiv L \mod 2$
- One says that ℓ is the *start index* and L als den *stop index* of the filter
- Orthogonality and low-pass properties of filters are expressed using

$$h(z) = \sum_{k=\ell}^{L} h_k\, z^k \quad \text{resp.} \quad H(\omega) = \sum_{k=\ell}^{L} h_k\, e^{i\omega} = h(e^{i\omega}).$$

Non-causal filters (2)

- The *orthogonality conditions* are again written as

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2,$$

- which is equivalent to

$$\sum_{k=\ell+2m}^{L} h_k h_{k-2m} = \delta_{m,0} \quad (0 \leq m < M)$$

Non-causal filters (3)

- If $\boldsymbol{g} = (g_\ell, \ldots, g_L)$ is another such filter with Fourier series $G(\omega)$, then the *orthogonality of* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ is written as

$$H(\omega) \cdot \overline{G(\omega)} + H(\omega + \pi) \cdot \overline{G(\omega + \pi)} = 0$$

- or equivalently

$$\sum_{k=\ell+2m}^{L} h_k \, g_{k-2m} = 0 \quad (0 \le m < M).$$

Non-causal filters (4)

- If $\boldsymbol{h}$ is an orthogonal filter, then $\boldsymbol{g}$ can be defined by

$$G(\omega) = e^{i(n\omega+b)}\,\overline{H(\omega+\pi)}$$

and this filter is automatically orthogonal

$$|G(\omega)|^2 + |G(\omega+\pi)|^2 = 2$$

- If n is any odd integer (and b any real number), then the reconstruction condition

$$H(\omega)\cdot\overline{G(\omega)} + H(\omega+\pi)\cdot\overline{G(\omega+\pi)} = 0$$

is also satisfied

Non-causal filters (5)

- Looking at filter coefficients, this means

$$g_k = -e^{i\,b}(-1)^k\, h_{n-k}$$

- Usually one takes $b = \pi$, so that this simplifies to

$$g_k = (-1)^k\, h_{n-k}$$

- In order to guarantee that $\boldsymbol{g}$ has start index ℓ and stop index L one has to take $n = L + \ell$

Coiflet filters (1)

- An obvious idea for constructing a low-pass filter $\boldsymbol{h} = (h_\ell, \ldots, h_L)$ is, apart from requiring orthogonality conditions

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$

and low-pass conditions at $\omega = \pi$, viz.,

$$H^{(m)}(\pi) = 0 \quad (m = 0, 1, 2, \ldots),$$

is to require low-pass conditions which specify the Fourier series $H(\omega)$ at $\omega = 0$

Coiflet filters (2)

- The condition

$$H(0) = \sqrt{2}$$

  is already satisfied
- In addition one may request for

$$H^{(m)}(0) = 0 \quad (m = 1, 2, \dots)$$

  which determine the behavior of $H(\omega)$ in the vicinity of $\omega = 0$, so that the values of the function are close to $\sqrt{2} = H(0)$
- This is the idea behind *Coiflet filters*, suggested by R. Coifman and realized by I. Daubechies (see the second one of the articles cited above)

Coiflet filters (3)

- The construction for these filters starts with the Daubechies polynomials

$$P_K(y) = \sum_{k=0}^{K} \binom{K+k}{k} y^k$$

- with their characteristic property

$$(*) \qquad (1-y)^K \cdot P_{K-1}(y) + y^K \cdot P_{K-1}(1-y) = 1$$

Coiflet filters (4)

- One makes an *Ansatz* for the Fourier series as

$$(**) \qquad H(\omega) = \sqrt{2}\,(1-y)^K \cdot \left[P_{K-1}(y) + y^K \cdot A(e^{i\omega})\right] \Big|_{y \leftarrow \sin^2(\omega/2)},$$

  where $A(z) = \sum_{k=0}^{2K-1} a_k\, z^k$ is to be a polynomial of degree $< 2K$

- From property $(*)$ one can write

$$(***) \qquad H(\omega) =$$

$$\sqrt{2} + \sqrt{2}\, y^K \cdot \left[-P_{K-1}(1-y) + (1-y)^K \cdot A(e^{i\omega})\right] \Big|_{y \leftarrow \sin^2(\omega/2)}$$

Coiflet filters (5)

- Looking at

$$(1-y)^K\Big|_{y\leftarrow\sin^2(\omega/2)} = \cos^{2K}(\omega/2) = \left[\frac{1}{2}e^{-i\omega/2}(1+e^{i\omega})\right]^{2K},$$

one realizes from $(**)$ that $H(\omega)$ has a root of multiplicity $2K$ for $\omega=\pi$:

$$H^{(m)}(\pi) = 0 \ (0 \le m < 2K)$$

Coiflet filters (6)

- Looking at

$$y^M\big|_{y\leftarrow\sin^2(\omega/2)} = \sin^{2M}(\omega/2) = \left[\frac{i}{2}e^{-i\omega/2}(1-e^{i\omega})\right]^{2M},$$

one realizes from $(***)$, that $H(\omega)-\sqrt{2}$ has a root of multiplicity $2K$ for $\omega = 0$:

$$H(0)=\sqrt{2} \quad \text{and} \quad H^{(m)}(0) = 0 \ (1 \leq m < 2K)$$

Coiflet filters (7)

- The previous assertions holds for any polynomial $A(z)$.
  The essential step is contained in the following claim
  (difficult, thus cited without proof) :
- The $2K$ coefficients $a_0, a_1, \ldots, a_{2K-1}$ of $A(z)$ can be chosen so that the orthogonality condition

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$

is satisfied

Coiflet filters (8)

- Now it must be clarified
  - how long the associated filter $\boldsymbol{h} = (h_\ell, \ldots, h_L)$ is
  - and what its start index ℓ and its stop index L are
- Write the right hand side of $(**)$ as a polynomial in $z = e^{i\omega}$.
  Reminder:

$$y = \sin^2(\omega/2) = \frac{1}{4}\left(2 - z - \frac{1}{z}\right)$$

$$1 - y = \cos^2(\omega/2) = \frac{1}{4}\left(2 + z + \frac{1}{z}\right)$$

Coiflet filters (9)

- Substituting in $(**)$ gives
  - $(1-y)^K$ has terms z^k for $-K \leq k \leq K$;
  - $P_{K-1}(1-y)$ has terms z^k for $-K+1 \leq k \leq K-1$;
  - y^K has terms z^k for $-K \leq k \leq K$;
  - $A(e^{i\omega})$ has terms z^k for $0 \leq k \leq 2K-1$
- The filter $H(\omega)$ specified by $(**)$ with parameter K
  - has start index $\ell = -2K$
  - and stop index $L = 4K-1$,
  - so that its length is $2M = 6K$
- This $\boldsymbol{h} = (h_{-2K}, \ldots, h_{4K-1})$ defines the *Coiflet* filter C_{6K}

Coiflet filters (10)

- For computing C_{6K} the following are relevant:
  - Orthogonality conditions

$$\sum_{k=-2K+2m}^{4K-1} h_k \, h_{k-2m} = \delta_{m,0} \qquad (0 \leq m < 3K)$$

  - Low-pass conditions

$$H^{(m)}(0) = 0 \quad (1 \leq m < 2K) \qquad\qquad H(0) = \sqrt{2}$$
$$H^{(m)}(\pi) = 0 \quad (0 \leq m < 2K)$$

- The orthogonal high-pass filter $G(\omega)$ which complements the low-pass filter $H(\omega)$ can be defined by

$$G(\omega) = e^{in+b} \cdot \overline{H(\omega + \pi)}$$

Coiflet-Filter C_6 (1)

- In case $K = 1$ the polynomial $A(z)$ has degree $2K - 1 = 1$. The *ansatz* for $H(\omega)$ resp. $h(z)$ then is

$$h(z) = \left(\frac{1}{2} + \frac{1}{4}\,z + \frac{1}{4}\,z^{-1}\right)\left(1 + \left(\frac{1}{2} - \frac{1}{4}\,z - \frac{1}{4}\,z^{-1}\right)(a_0 + a_1 z)\right)$$
$$= (-\frac{1}{16}\,a_0 z^{-2} + \left(-\frac{1}{16}\,a_1 + \frac{1}{4}\right) z^{-1} + \frac{1}{8}\,a_0 + \frac{1}{2}$$
$$+ \left(\frac{1}{8}\,a_1 + \frac{1}{4}\right) z - \frac{1}{16}\,a_0 z^2 - \frac{1}{16}\,a_1 z^3)$$

- Thus $\boldsymbol{h} = (h_{-2}, \ldots, h_3)$ with the coefficients a_0, a_1 to be determined is given by

$$\sqrt{2} \cdot [-\frac{1}{16}\,a_0, \left(-\frac{1}{16}\,a_1 + \frac{1}{4}\right), \left(\frac{1}{8}\,a_0 + \frac{1}{2}\right), \left(\frac{1}{8}\,a_1 + \frac{1}{4}\right), -\frac{1}{16}\,a_0, -\frac{1}{16}\,a_1]$$

Coiflet filter C_6 (2)

- The orthogonality condition $\sum_k h_k^2 = 1$ gives

$$3\,{a_0}^2 + 3\,{a_1}^2 + 4\,a_1 + 48 + 16\,a_0 = 64$$

- The orthogonality condition $\sum_k h_k h_{k+2} = 0$ gives

$$-{a_0}^2 - 4\,a_0 - {a_1}^2 + 4 = 0$$

- The orthogonality condition $\sum_k h_k h_{k+4} = 0$ gives

$$ {a_0}^2 + {a_1}^2 - 4\,a_1 = 0$$

- The solution of these three equations is

$$a_0 = 1 - \alpha,\ \ a_1 = \alpha = \sqrt{1 - 6\,z + 2\,z^2}$$

Coiflet-Filter C_6 (3)

- This leads to

$$h = \sqrt{2} \cdot \left[-\frac{1-\alpha}{16}, -\frac{\alpha}{16} + \frac{1}{4}, \frac{5}{8} - \frac{\alpha}{8}, \frac{\alpha}{8} + \frac{1}{4}, -\frac{1-\alpha}{16}, -\frac{\alpha}{16} \right]$$

- and floating-point approximations of the filter coefficients are

h_{-2}	-0.0727326195
h_{-1}	0.3378976624
h_0	0.8525720199
h_1	0.3848648468
h_2	-0.0727326195
h_3	-0.0156557281

Coiflet filter C_6 (4)

- Since start and stop indices of the filter are known ($\ell = -2$ and $L = 3$), one many make an *ansatz* for $\boldsymbol{h} = (h_{-2}, \ldots, h_3)$ with undetermined coefficients and try to solve
- the three orthogonality conditions

$$\sum_k h_k^2 = 1 \qquad \sum_k h_k h_{k+2} = 0 \qquad \sum_k h_k h_{k+4} = 0$$

- and the four low-pass conditions

$$H(0) = \sqrt{2} \qquad H(\pi) = 0 \qquad H'(0) = 0 \qquad H'(\pi) = 0$$

directly

Coiflet filter C_6 (5)

- The following are the relevant equations:

$$
\begin{aligned}
h_{-2}^2 + h_{-1}^2 + h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\
h_{-2}h_0 + h_{-1}h_1 + h_0h_2 + h_1h_3 &= 0 \\
h_{-2}h_2 + h_{-1}h_3 &= 0 \\
h_{-2} + h_{-1} + h_0 + h_1 + h_2 + h_3 &= \sqrt{2} \\
h_{-2} - h_{-1} + h_0 - h_1 + h_2 - h_3 &= 0 \\
-2\,h_{-2} - h_{-1} + h_1 + 2\,h_2 + 3\,h_3 &= 0 \\
2\,h_{-2} - h_{-1} + h_1 - 2\,h_2 + 3\,h_3 &= 0
\end{aligned}
$$

Coiflet filter C_6 (6)

- This gives two solutions:

$$\begin{bmatrix} h_{-2} & \frac{1}{32}\sqrt{2}+\frac{1}{32}\sqrt{14} & \frac{1}{32}\sqrt{2}-\frac{1}{32}\sqrt{14} \\ h_{-1} & \frac{5}{32}\sqrt{2}-\frac{1}{32}\sqrt{14} & \frac{5}{32}\sqrt{2}+\frac{1}{32}\sqrt{14} \\ h_0 & \frac{7}{16}\sqrt{2}-\frac{1}{16}\sqrt{14} & \frac{7}{16}\sqrt{2}+\frac{1}{16}\sqrt{14} \\ h_1 & \frac{7}{16}\sqrt{2}+\frac{1}{16}\sqrt{14} & \frac{7}{16}\sqrt{2}-\frac{1}{16}\sqrt{14} \\ h_2 & \frac{1}{32}\sqrt{2}+\frac{1}{32}\sqrt{14} & \frac{1}{32}\sqrt{2}-\frac{1}{32}\sqrt{14} \\ h_3 & -\frac{3}{32}\sqrt{2}-\frac{1}{32}\sqrt{14} & -\frac{3}{32}\sqrt{2}+\frac{1}{32}\sqrt{14} \end{bmatrix}$$

Coiflet-Filter C_6 (7)

- and the floating-point approximation is

h_{-2}	0.1611209671	−0.07273261949
h_{-1}	0.1040440758	0.3378976624
h_0	0.3848648467	0.8525720201
h_1	0.8525720201	0.3848648467
h_2	0.1611209671	−0.07273261949
h_3	−0.2495093147	−0.0156557281

Coiflet filter C_{12} (1)

- In the case $K = 2$ one looks for a filter with start index $\ell = -4$ and stop index $L = 7$
- Proceeding as in the previous section leads to the following system of equations for the filter coefficients $h_{-4}, \ldots, h_7$:

Coiflet filter C_{12} (2)

The orthogonality conditions

$$\begin{aligned}
h_{-4}^2 + h_{-3}^2 + h_{-2}^2 + h_{-1}^2 + h_0^2 + h_1^2 + h_2^2 + h_3^2 + h_4^2 + h_5^2 + h_6^2 + h_7^2 &= 1 \\
h_{-4}h_{-2} + h_{-3}h_{-1} + h_{-2}h_0 + h_{-1}h_1 + h_0h_2 + h_1h_3 + h_2h_4 + h_3h_5 + h_4h_6 + h_5h_7 &= 0 \\
h_{-4}h_0 + h_{-3}h_1 + h_{-2}h_2 + h_{-1}h_3 + h_0h_4 + h_1h_5 + h_2h_6 + h_3h_7 &= 0 \\
h_{-4}h_2 + h_{-3}h_3 + h_{-2}h_4 + h_{-1}h_5 + h_0h_6 + h_1h_7 &= 0 \\
h_{-4}h_4 + h_{-3}h_5 + h_{-2}h_6 + h_{-1}h_7 &= 0 \\
h_{-4}h_6 + h_{-3}h_7 &= 0
\end{aligned}$$

Coiflet filter C_{12} (3)

The low-pass conditions

$$\begin{aligned}
h_{-4} + h_{-3} + h_{-2} + h_{-1} + h_0 + h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7 &= \sqrt{2} \\
h_{-4} - h_{-3} + h_{-2} - h_{-1} + h_0 - h_1 + h_2 - h_3 + h_4 - h_5 + h_6 - h_7 &= 0 \\
4h_{-4} + 3h_{-3} + 2h_{-2} + h_{-1} - h_1 - 2h_2 - 3h_3 - 4h_4 - 5h_5 - 6h_6 - 7h_7 &= 0 \\
4h_{-4} - 3h_{-3} + 2h_{-2} - h_{-1} + h_1 - 2h_2 + 3h_3 - 4h_4 + 5h_5 - 6h_6 + 7h_7 &= 0 \\
-16h_{-4} - 9h_{-3} - 4h_{-2} - h_{-1} - h_1 - 4h_2 - 9h_3 - 16h_4 - 25h_5 - 36h_6 - 49h_7 &= 0 \\
-16h_{-4} + 9h_{-3} - 4h_{-2} + h_{-1} + h_1 - 4h_2 + 9h_3 - 16h_4 + 25h_5 - 36h_6 + 49h_7 &= 0 \\
-64h_{-4} - 27h_{-3} - 8h_{-2} - h_{-1} + h_1 + 8h_2 + 27h_3 + 64h_4 + 125h_5 + 216h_6 + 343h_7 &= 0 \\
-64h_{-4} + 27h_{-3} - 8h_{-2} + h_{-1} - h_1 + 8h_2 - 27h_3 + 64h_4 - 125h_5 + 216h_6 - 343h_7 &= 0
\end{aligned}$$

Coiflet filter C_{12} (4)

The 8 low-pass conditions are linear, so one first uses these in order to eliminate 8 out of 12 variables:

$$
\begin{aligned}
h_{-4} &= 4\,h_6 + h_4 \\
h_{-3} &= h_5 + 4\,h_7 - 1/32\,\sqrt{2} \\
h_{-2} &= -15\,h_6 - 4\,h_4 \\
h_{-1} &= -4\,h_5 - 15\,h_7 + \frac{9}{32}\,\sqrt{2} \\
h_0 &= 20\,h_6 + 6\,h_4 + 1/2\,\sqrt{2} \\
h_1 &= 6\,h_5 + 20\,h_7 + \frac{9}{32}\,\sqrt{2} \\
h_2 &= -10\,h_6 - 4\,h_4 \\
h_3 &= -4\,h_5 - 10\,h_7 - 1/32\,\sqrt{2}
\end{aligned}
$$

Coiflet filter C_{12} (5)

It remains to solve the following <u>non-linear</u> system of equations:

$$\frac{21\sqrt{2}}{16}\,h_5 + \frac{51\sqrt{2}}{16}\,h_7 + 20\sqrt{2}\,h_6 + 6\sqrt{2}\,h_4 + 448\,h_5 h_7 + 448\,h_4 h_6 + 742\,h_7^2 + 70\,h_4^2 + 70\,h_5^2 + 742\,h_6^2 = \frac{23}{128}$$

$$-\frac{3\sqrt{2}}{8}\,h_5 - \frac{7\sqrt{2}}{16}\,h_7 - \frac{25\sqrt{2}}{2}\,h_6 - 4\sqrt{2}\,h_4 - 350\,h_5 h_7 - 350\,h_4 h_6 - 560\,h_7^2 - 56\,h_4^2 - 56\,h_5^2 - 560\,h_6^2 = \frac{63}{512}$$

$$-\frac{5\sqrt{2}}{8}\,h_5 - \frac{15\sqrt{2}}{8}\,h_7 + 2\sqrt{2}\,h_6 + \sqrt{2}\,h_4 + 160\,h_5 h_7 + 160\,h_4 h_6 + 220\,h_7^2 + 28\,h_4^2 + 28\,h_5^2 + 220\,h_6^2 = \frac{9}{256}$$

$$-20\,h_6^2 - 35\,h_4 h_6 - 8\,h_4^2 - 8\,h_5^2 - 35\,h_5 h_7 + \frac{3\sqrt{2}}{8}\,h_5 - 20\,h_7^2 + \frac{15\sqrt{2}}{32}\,h_7 + \frac{\sqrt{2}}{2}\,h_6 = \frac{1}{512}$$

$$h_4^2 + h_5^2 - \frac{\sqrt{2}}{32}\,h_5 - 15\,h_6^2 - 15\,h_7^2 + \frac{9\sqrt{2}}{32}\,h_7 = 0$$

$$4\,h_6^2 + h_4 h_6 + h_5 h_7 + 4\,h_7^2 - \frac{\sqrt{2}}{32}\,h_7 = 0$$

Coiflet-Filter C_{12} (6)

The solution turns out to be

$$h_4 = -\frac{1}{1024}\,\frac{1430\,\alpha^3 + 5064\,\sqrt{2}\,\alpha^2 + 10441\,\alpha + 2590\,\sqrt{2}}{338\,\alpha^2 + 962\,\sqrt{2}\,\alpha + 1369}$$

$$h_5 = \frac{1}{2048}\,\frac{1615\,\sqrt{2}\,\alpha + 4081 + 65\,\alpha^2}{26\,\alpha + 37\,\sqrt{2}}$$

$$h_6 = \frac{1}{1024}\,\alpha$$

$$h_7 = -\frac{1}{2048}\,\frac{179\,\sqrt{2}\,\alpha + 405 + 21\,\alpha^2}{26\,\alpha + 37\sqrt{2}}$$

where α is a solution of the degree 4 polynomial equation

$$25Z^4 - 1082\,\sqrt{2}\,Z^3 - 32180\,Z^2 - 77370\,\sqrt{2}\,Z - 102375 = 0,$$

so that one expects 4 distinct solutions

Coiflet-Filter C_{12} (7)

Here are the solutions in floating-point approximation:

$$\begin{bmatrix}
-0.00135879906 & -0.02881077935 & 0.01638733604 & -0.0216835830 \\
-0.01461155251 & 0.00954232518 & -0.04146493789 & -0.04759942451 \\
-0.0074103835 & 0.1131648994 & -0.06737255304 & 0.163253958 \\
0.2806116518 & 0.1765268828 & 0.3861100713 & 0.3765105895 \\
0.7503363057 & 0.5425549768 & 0.8127236327 & 0.2709267760 \\
0.5704650013 & 0.7452653006 & 0.4170051772 & 0.5167479708 \\
-0.0716382822 & 0.1027738095 & -0.07648859743 & 0.5458520919 \\
-0.1553572228 & -0.2967882834 & -0.05943441354 & -0.2397210372 \\
0.05002351996 & -0.02049790739 & 0.02368017155 & -0.3277620898 \\
0.02480433052 & 0.07883524141 & 0.005611433291 & 0.1360266602 \\
-0.01284557976 & -0.002078217989 & -0.001823208878 & 0.07651962671 \\
0.001194572696 & -0.006274685605 & -0.0007205493428 & -0.03485797772
\end{bmatrix}$$

The third column of this matrix is what is usually taken as the Coiflet filter of length 12