Motivating Bi-Orthogonality (alias dual bases)

- A 3D-example
  - The standard ON-basis of $\mathbb{R}^3$

$$\mathcal{E} = \{\boldsymbol{e}^1 = (1,0,0), \boldsymbol{e}^2 = (0,1,0), \boldsymbol{e}^3 = (0,0,1)\}$$

  - Another basis of $\mathbb{R}^3$

$$\mathcal{B} = \{\boldsymbol{b}^1 = (1,0,1), \boldsymbol{b}^2 = (0,-1,1), \boldsymbol{b}^3 = (1,0,2)\}$$

  - In matrix form

$$B = \left[\langle \boldsymbol{b}^i \mid \boldsymbol{e}^j \rangle\right]_{1 \le i,j \le 3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

  - The inverse of B

$$B^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

- The basis $\mathcal{C} = \{\boldsymbol{c}^1, \boldsymbol{c}^2, \boldsymbol{c}^3\}$ defined by the rows of

$$C = \left[\langle \boldsymbol{c}^i \mid \boldsymbol{e}^j \rangle\right]_{1 \le i,j \le 3} = (B^{-1})^t = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

  is $\mathcal{C} = \{\boldsymbol{c}^1 = (2, -1, -1), \boldsymbol{c}^2 = (0, -1, 0), \boldsymbol{c}^3 = (-1, 1, 1)\}$
- $\mathcal{B}$ and $\mathcal{C}$ are *bi-orthogonal* (or *dual bases*) in the sense that

$$\langle \boldsymbol{b}^i \mid \boldsymbol{c}^j \rangle = \delta_{i,j}$$

- in particular

$$\boldsymbol{c}^1 \perp \text{the plane spanned by } \boldsymbol{b}^2 \text{ and } \boldsymbol{b}^3$$
$$\boldsymbol{c}^2 \perp \text{the plane spanned by } \boldsymbol{b}^1 \text{ and } \boldsymbol{b}^3$$
$$\boldsymbol{c}^3 \perp \text{the plane spanned by } \boldsymbol{b}^1 \text{ and } \boldsymbol{b}^2$$

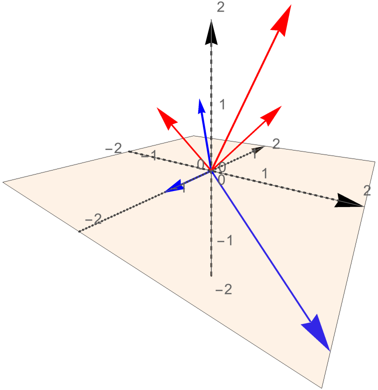

Figure: The standard ON-basis $\mathcal{E}$, the basis $\mathcal{B}$ (red) and the basis $\mathcal{C}$ (blue)

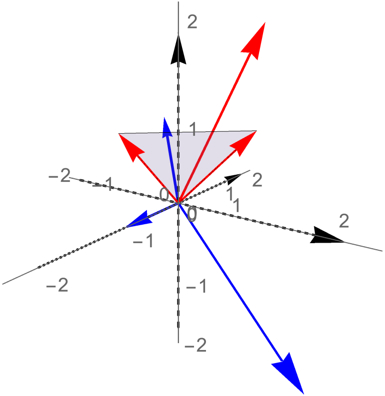

Figure: The plane spanned by $\{\boldsymbol{b}^1, \boldsymbol{b}^2\}$ (grey)

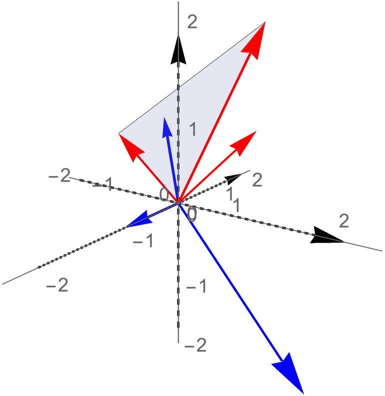

Figure: The plane spanned by $\{\boldsymbol{b}^2, \boldsymbol{b}^3\}$

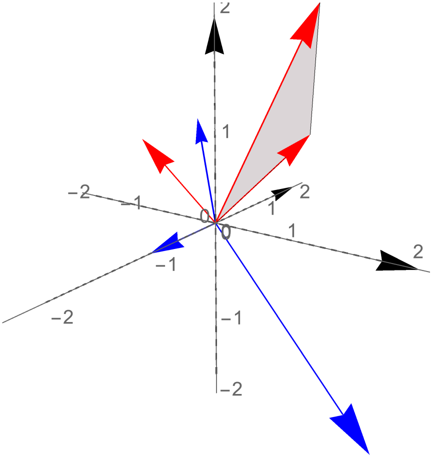

Figure: The plane spanned by $\{\boldsymbol{b}^1, \boldsymbol{b}^3\}$

- Representing a vector in bases $\mathcal{B}$ and $\mathcal{C}$

$$
\begin{aligned}
\boldsymbol{v} = (-1,1,1) &= (2,1,-1).B \\
&= \underbrace{\langle \boldsymbol{v} \mid \boldsymbol{c}^1 \rangle}_{=2} \boldsymbol{b}^1 + \underbrace{\langle \boldsymbol{v} \mid \boldsymbol{c}^2 \rangle}_{=1} \boldsymbol{b}^2 + \underbrace{\langle \boldsymbol{v} \mid \boldsymbol{c}^3 \rangle}_{-1} \boldsymbol{b}^3 \\
&= (2,2,3).C \\
&= \underbrace{\langle \boldsymbol{v} \mid \boldsymbol{b}^1 \rangle}_{=2} \boldsymbol{c}^1 + \underbrace{\langle \boldsymbol{v} \mid \boldsymbol{b}^2 \rangle}_{=2} \boldsymbol{c}^2 + \underbrace{\langle \boldsymbol{v} \mid \boldsymbol{b}^3 \rangle}_{=3} \boldsymbol{c}^3
\end{aligned}
$$

- Warning: Dual bases need not to be "nice"
- Example

$$B = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \qquad B^{-1} = \frac{1}{13}\begin{bmatrix} -2 & 4 & 3 \\ -7 & 1 & 4 \\ 1 & -2 & 5 \end{bmatrix}$$

- Example

$$B = \begin{bmatrix} 2 & -5 & 5 & 2 \\ 5 & 5 & 2 & 4 \\ 1 & -4 & 0 & 4 \\ 2 & 0 & -5 & -5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{1513}\begin{bmatrix} 140 & 160 & 25 & 204 \\ -106 & 95 & -127 & -68 \\ 197 & 9 & -235 & -102 \\ -141 & 55 & 245 & -119 \end{bmatrix}$$

- V a finite-dimensional real or complex vector space with inner product $\langle\, . \,|\, . \,\rangle$ and an ON basis $\mathcal{E} = (\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n)$, i.e.,

$$\langle\, \boldsymbol{e}^i \,|\, \boldsymbol{e}^j \,\rangle = \delta_{i,j} \qquad (1 \leq i, j \leq n)$$

- Orthogonality: For any vector $\boldsymbol{v} \in V$

$$\boldsymbol{v} = \sum_{i=1}^{n} \varepsilon_i \, \boldsymbol{e}^i \quad \text{with} \quad \varepsilon_i = \langle\, v \,|\, \boldsymbol{e}^i \,\rangle$$

- Now let $\mathcal{B} = (\boldsymbol{b}^1, \ldots, \boldsymbol{b}^n)$ be an arbitrary basis of V, not necessarily orthogonal. The any $\boldsymbol{v} \in V$ can be written as

$$\boldsymbol{v} = \sum_{i=1}^{n} \beta_i \, \boldsymbol{b}^i$$

But what are the coefficients β_i in terms of $\langle\, . \,|\, . \,\rangle$?

- With $\mathcal{E}$ and $\mathcal{B}$ as before, let

$$\boldsymbol{b}^i = \sum_{j=1}^{n} b_{i,j}\, \boldsymbol{e}^j, \quad \text{where} \quad b_{i,j} = \langle\, \boldsymbol{b}^i \mid \boldsymbol{e}^j \,\rangle$$

$$B = \left[b_{i,j}\right]_{1 \le i,j \le n}$$

  Then B is an invertible matrix, because $\mathcal{B}$ is a basis
- Now let $\mathcal{C} = (\boldsymbol{c}^1, \ldots, \boldsymbol{c}^n)$ be another basis of V with coefficient matrix $C = \left[C_{i,j}\right]_{1 \le i,j \le n}$ and $c_{i,j} = \langle\, \boldsymbol{c}^i \mid \boldsymbol{e}^j \,\rangle$
- $\mathcal{B}$ and $\mathcal{C}$ are said to be a *bi-orthogonal pair of bases* (or *dual bases*) if

$$\langle\, \boldsymbol{b}^i \mid \boldsymbol{c}^j \,\rangle = \delta_{i,j} \quad (1 \le i, j \le n)$$

- $\boldsymbol{Q}_1$: Does such a dual basis always exist? Is it unique?
- $\boldsymbol{Q}_2$: What is the benefit of this?

- Answer to $\boldsymbol{Q}_2$:

  Assume that $\mathcal{B}$ and $\mathcal{C}$ are a bi-orthogonal pair of bases of V. For $\boldsymbol{v} \in V$ write

  $$\boldsymbol{v} = \sum_{i=1}^{n} \beta_i \, \boldsymbol{b}^i = \sum_{j=1}^{n} \gamma_j \, \boldsymbol{c}^j$$

  Then

  $$\langle \, \boldsymbol{v} \mid \boldsymbol{c}^j \, \rangle = \sum_{i=1}^{n} \beta_i \, \langle \, \boldsymbol{b}^i \mid \boldsymbol{c}^j \, \rangle = \beta_j$$

  and

  $$\langle \, \boldsymbol{v} \mid \boldsymbol{b}^i \, \rangle = \sum_{j=1}^{n} \gamma_j \, \langle \, \boldsymbol{c}^j \mid \boldsymbol{b}^i \, \rangle = \gamma_i$$

  Hence

  $$\boldsymbol{v} = \sum_{i=1}^{n} \langle \, \boldsymbol{v} \mid \boldsymbol{c}^i \, \rangle \, \boldsymbol{b}^i = \sum_{j=1}^{n} \langle \, \boldsymbol{v} \mid \boldsymbol{b}^j \, \rangle \, \boldsymbol{c}^j$$

- Answer to $\boldsymbol{Q}_1$:

  Let $\mathcal{B}$ and $\mathcal{C}$ be any bases of V, as above,
  then for any $1 \leq i, k \leq n$ because of the orthonormality of $\mathcal{E}$:

$$\begin{aligned}\langle\, \boldsymbol{b}^i \mid \boldsymbol{c}^k \,\rangle &= \langle\, \sum_{j=1}^{n} b_{i,j}\, \boldsymbol{e}^j \mid \sum_{\ell=1}^{n} c_{k,\ell}\, \boldsymbol{e}^\ell \,\rangle \\ &= \sum_{j=1}^{n} \sum_{\ell=1}^{n} b_{i,j}\, \overline{c_{k,\ell}}\, \langle\, \boldsymbol{e}^j \mid \boldsymbol{e}^\ell \,\rangle = \sum_{j=1}^{n} b_{i,j}\, \overline{c_{k,j}}\end{aligned}$$

  In matrix terms $\left[\langle\, \boldsymbol{b}^i \mid \boldsymbol{c}^k \,\rangle\right]_{1 \leq i,k \leq n} = B \cdot C^\dagger$

  Recall: $C^\dagger$ is the conjugate-transpose of C, also called the *adjoint* of C

  Hence $\mathcal{B}$ and $\mathcal{C}$ are dual bases $\leftrightarrow B \cdot C^\dagger = I_n \leftrightarrow B^{-1} = C^\dagger$
  which guarantees existence and uniqueness