Constructing biorthogonal filter pairs

WTBV

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Bspline filters of odd length $N+1 = 2\ell + 1$

- Frequency representation

$$H(\omega) = \sqrt{2}\cos^N(\omega/2)$$

- Filter coefficients

$$\begin{aligned}\boldsymbol{h} &= (h_{-\ell}, h_{-\ell+1}, \ldots, h_\ell) \\ &= \frac{\sqrt{2}}{2^N}\left(\binom{N}{0}, \binom{N}{1}, \ldots, \binom{N}{N}\right)\end{aligned}$$

- This is a symmetric low-pass filter

$$H(0) = \sqrt{2}, H(\pi) = H'(\pi) = \ldots = H^{(N-1)}(\pi) = 0$$

- Daubechies polynomials

$$P_M(z) = \sum_{m=0}^{M} \binom{M+m}{m} z^m$$

- The "Bézout property" of the Daubechies polynomials

$$(1-z)^{M+1} P_M(z) + z^{M+1} P_M(1-z) = 1$$

Biorthogonal partners of the Bspline filters

- $\boldsymbol{h}$: Bspline filter of odd length $N+1=2\ell+1$, as before, with

$$H(\omega)=\sqrt{2}\cos^{N}(\omega/2)$$

- $\tilde{N}+1=2\tilde{\ell}+1$, so that $2\tilde{N}+N-1$ is odd
- $\tilde{\boldsymbol{h}}$: a filter defined by its frequency representation

$$\tilde{H}(\omega)=\sqrt{2}\cos^{\tilde{N}}(\omega/2)\cdot P_{\ell+\tilde{\ell}-1}(\sin^{2}(\omega/2))$$

- Filter coefficients

$$\tilde{\boldsymbol{h}}=\left(h_{-2\tilde{\ell}-\ell+1},h_{-2\tilde{\ell}-\ell+2},\ldots,h_{2\tilde{\ell}+\ell-1}\right)$$

- The filter $\tilde{\boldsymbol{h}}$
  - has length $\tilde{N}+2(\ell+\tilde{\ell}-1)+1=2\tilde{N}+N-1$
  - is symmetric
  - is orthogonal to $\boldsymbol{h}$
  - But is it a good low-pass filter?

- Orthogonality of $(\boldsymbol{h}, \tilde{\boldsymbol{h}})$, i.e.

$$\tilde{H}(\omega)\,\overline{H(\omega)} + \tilde{H}(\omega+\pi)\,\overline{H(\omega+\pi)} = 2$$

  follows from

$$H(\omega)\,\tilde{H}(\omega) = 2\cos^{N+\tilde{N}}(\omega/2)\,P_{\ell+\tilde{\ell}-1}(\sin^2(\omega/2))$$

  by making use of properties of the Daubechies polynomials, in particular the "Bézout property" mentioned before
- Notabene: a similar argument can be made to construct orthogonal pairs of symmetric filters of even length
- Notation: $K_{M,N}$ is the bi-orthogonal partner filter $\tilde{\boldsymbol{h}}$ of length $2M+N-1$ of a Bspline filter $\boldsymbol{h}$ of length $N+1$

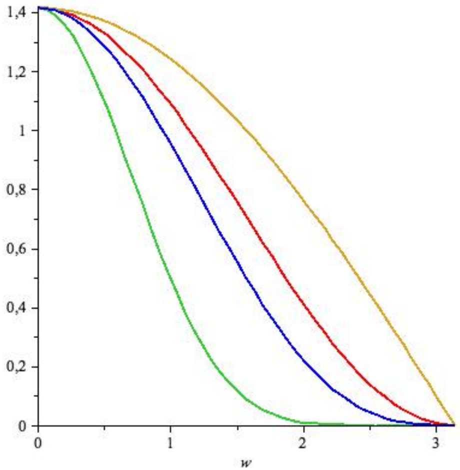

Figure: Frequency representations of the Bspline filters of length 2,3,4,8

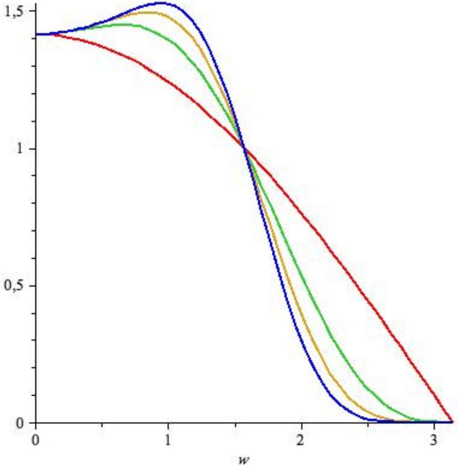

Figure: Bspline filter partners $K_{1,1}, K_{3,1}, K_{5,1}, K_{7,1}$

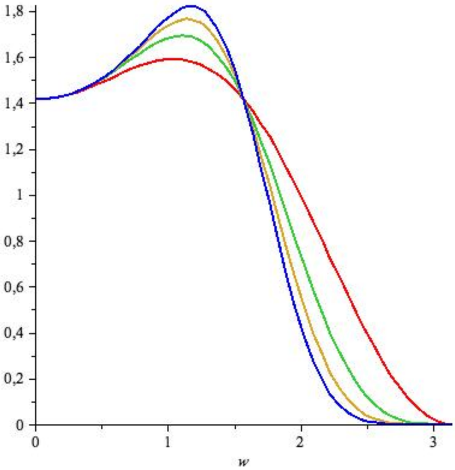

Figure: Bspline filter partners $K_{2,2}, K_{2,4}, K_{2,6}, K_{2,8}$

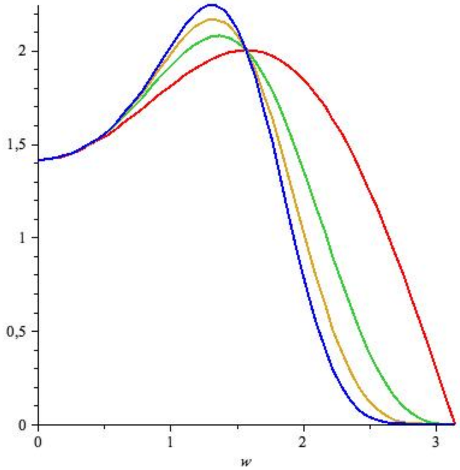

Figure: Bspline filter partners $K_{1,3}, K_{3,3}, K_{5,3}, K_{7,3}$

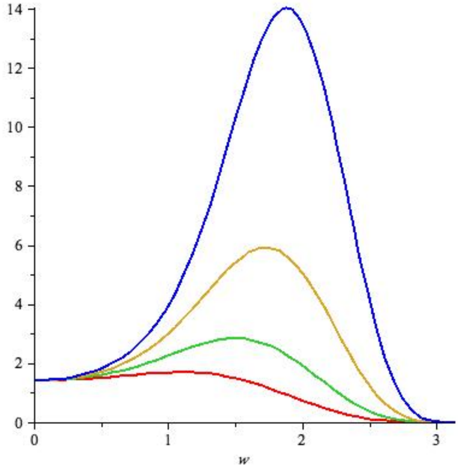

Figure: Bspline filter partners $K_{4,2}, K_{4,4}, K_{4,6}, K_{4,8}$

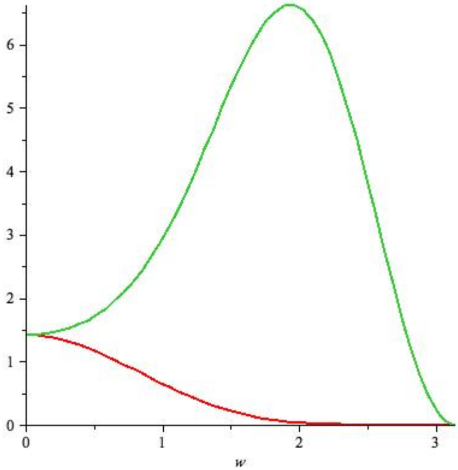

Figure: (7,9) Bspline filter pair

A better idea (Cohen-Daubechies-Feauvau)

- The properties of the product

$$H(\omega) \cdot \tilde{H}(\omega) = \sqrt{2}\cos^{2\ell}(\omega/2) \cdot \sqrt{2}\cos^{2\tilde{\ell}}(\omega/2) \cdot P_{\ell+\tilde{\ell}-1}(\sin^2(\omega/2))$$
$$= 2\cos^{2(\ell+\tilde{\ell})}(\omega/2) \cdot P_{\ell+\tilde{\ell}-1}(\sin^2(\omega/2))$$

are instrumental in proving orthogonality

- Thus any factorization of

$$2\cos^{2(\ell+\tilde{\ell})}(\omega/2) \cdot P_{\ell+\tilde{\ell}-1}(\sin^2(\omega/2))$$

into two factors

$$\underbrace{\sqrt{2}\cos^{2\ell}(\omega/2) \cdot p(\cos(\omega))}_{H(\omega)} \cdot \underbrace{\sqrt{2}\cos^{2\tilde{\ell}}(\omega/2) \cdot \tilde{p}(\cos(\omega))}_{\tilde{H}(\omega)}$$

with polynomials $p, \tilde{p}$ s.th.

$$p(\cos(\omega)) \cdot \tilde{p}(\cos(\omega)) = P_{\ell+\tilde{\ell}-1}(\sin^2(\omega/2))$$

would give a pair of biorthogonal filters

- The factorization considered so far is too unbalanced

Constructing the Cohen-Daubechies-Feauveau-7/9 filter pair

- Start with the Daubechies polynomial

$$P_3(z) = \binom{3}{0} + \binom{4}{1} z + \binom{5}{2} z^2 + \binom{6}{3} z^3 = 1 + 4z + 10z^2 + 20z^3$$

- The 3 complex roots of this polynomial can be determined exactly

$$z_1 = \frac{1}{6}\left(-1 - \frac{7^{2/3}}{\sqrt[3]{5\left(3\sqrt{15}-10\right)}} + \frac{\sqrt[3]{7\left(3\sqrt{15}-10\right)}}{5^{2/3}}\right)$$

$$z_2 = -\frac{1}{6} + \frac{7^{2/3}\left(1+i\sqrt{3}\right)}{12\sqrt[3]{5\left(3\sqrt{15}-10\right)}} - \frac{\left(1-i\sqrt{3}\right)\sqrt[3]{7\left(3\sqrt{15}-10\right)}}{12\ 5^{2/3}}$$

$$z_3 = -\frac{1}{6} + \frac{7^{2/3}\left(1-i\sqrt{3}\right)}{12\sqrt[3]{5\left(3\sqrt{15}-10\right)}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{7\left(3\sqrt{15}-10\right)}}{12\ 5^{2/3}}$$

- It suffices to take approximate values

$$
\begin{aligned}
z_1 &\approx -0.342384 \\
z_2 &\approx -0.078808 + 0.373931i \\
z_3 &\approx -0.078808 - 0.373931i
\end{aligned}
$$

- The polynomial $P_3(z)$ factors into two polynomials

$$
\begin{aligned}
p(z) &= a \cdot (z - z_1) \\
\widetilde{p}(z) &= \frac{1}{a} \cdot (z - z_2)(z - z_3)
\end{aligned}
$$

where the constant a has to be determined

- In terms of approximate values

$$
\begin{aligned}
p(z) &\approx a \cdot (z + 0.342384) \\
\widetilde{p}(z) &\approx \frac{1}{a}(z + 0.078808 - 0.373931i)(z + 0.078808 + 0.373931i) \\
&\approx \frac{1}{a}(2.9207 + 3.15232z + 20z^2)
\end{aligned}
$$

- The two filters $\boldsymbol{h} = (h_j)_{j=-3..3}$ and $\widetilde{\boldsymbol{h}} = (\widetilde{h}_j)_{j=-4..4}$ are defined through their frequency representations (note that $K = 4, \ell = \widetilde{\ell} = 2$)

$$\begin{aligned} H(\omega) &= \sqrt{2}\cos(\omega/2)^4 p(\sin(\omega/2)^2) \\ &= a \cdot \sqrt{2}\cos(\omega/2)^4 \left(0.342384 + \sin(\omega/2)^2\right) \\ \widetilde{H}(\omega) &= \sqrt{2}\cos(\omega/2)^4 \widetilde{p}(\sin(\omega/2)^2) \\ &= \frac{1}{a}\cos(\omega/2)^4\left(4.13049 + 4.45805\sin(\omega/2)^2 + 20\sqrt{2}\sin(\omega/2)^4\right) \end{aligned}$$

- Now the value of a can be fixed by requiring $H(0) = \sqrt{2}$ (and also $\widetilde{H}(0) = \sqrt{2}$), which gives

$$a = 2.9207$$

- so that

$$\begin{aligned} H(\omega) &= 4.13049\cos(\omega/2)^4\left(0.342384 + \sin(\omega/2)^2\right) \\ \widetilde{H}(\omega) &= \cos(\omega/2)^4\left(1.41421 + 1.52637\sin(\omega/2)^2 + 9.68408\sin(\omega/2)^4\right) \end{aligned}$$

- Converting the sin- and cos-expressions into exponentials then gives the filter coefficients

$$(h_j)_{j=-3..3} = \begin{bmatrix} -0.0645388826 \\ -0.0406894175 \\ 0.4180922731 \\ 0.7884856164 \\ 0.4180922731 \\ -0.0406894175 \\ -0.0645388826 \end{bmatrix} \quad (\widetilde{h}_j)_{j=-4..4} = \begin{bmatrix} 0.0378284555 \\ -0.0238494650 \\ -0.1106244044 \\ 0.3774028555 \\ 0.8526986788 \\ 0.3774028555 \\ -0.1106244044 \\ -0.0238494650 \\ 0.0378284555 \end{bmatrix}$$

- Low-pass properties: from the definition it is clear that both filters $\boldsymbol{h} = (h_j)_{j=-3..3}$ and $\widetilde{\boldsymbol{h}} = (\widetilde{h}_j)_{j=-4..4}$ have 4 vanishing moments, i.e., they have very good smoothness properties for reconstruction

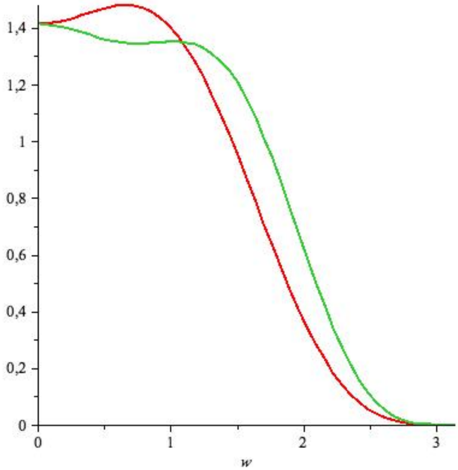

Figure: Frequency picture of the Cohen-Daubechies-Feauveau-(7,9) filter pair

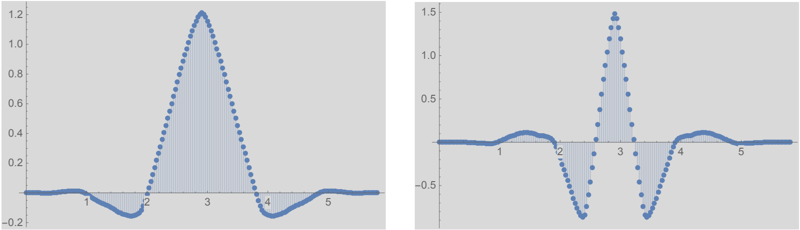

Figure: Scaling and wavelet functions for the CDF-7 filter

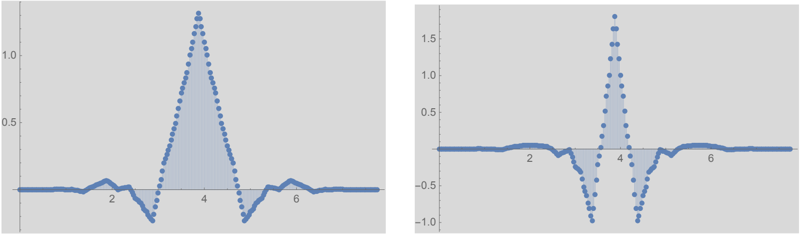

Figure: Scaling and wavelet functions for the CDF-9 filter