The HAAR Wavelet Transform

WTBV-WS17/18

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Definitions (1)

- Dyadic intervals $(j, k \in \mathbb{Z})$

$$\begin{aligned} I_{j,k} &= [\, k/2^j,\, (k+1)/2^j \,) \\ &= I_{j+1,2k} \uplus I_{j+1,2k+1} \end{aligned}$$

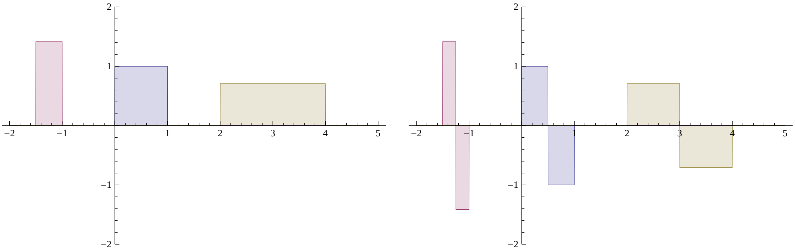
- HAAR functions

$$\begin{aligned} &\text{HAAR scaling function} & \phi(t) &= \mathbf{1}_{[0,1)}(t) \\ &\text{HAAR wavelet function} & \psi(t) &= \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t) \\ & & &= \phi(2t) - \phi(2t+1) \end{aligned}$$

- Dilation and translation of HAAR functions $(j, k \in \mathbb{Z})$:

$$\begin{aligned} \phi_{j,k}(t) &= 2^{j/2}\,\mathbf{1}_{I_{j,k}}(t) = 2^{j/2}\,\phi(2^j t - k) = (D_{2^j} T_k \phi)(t) \\ \psi_{j,k}(t) &= 2^{j/2}\,\psi(2^j t - k) = (D_{2^j} T_k \psi)(t) \end{aligned}$$

Examples:



left: HAAR scaling functions $\phi_{1,-3}, \phi_{0,0}, \phi_{-1,1}$

right: HAAR wavelet functions $\psi_{1,-3}, \psi_{0,0}, \psi_{-1,1}$

Definitions (2)

- The following families of functions are of interest:

$$\Phi = \{\phi_{j,k}\}_{j,k\in\mathbb{Z}} : \text{HAAR scaling functions (all levels)}$$

$$\Phi_j = \{\phi_{j,k}\}_{k\in\mathbb{Z}} : \text{HAAR scaling functions on level } j$$

$$\Psi_j = \{\psi_{j,k}\}_{k\in\mathbb{Z}} : \text{HAAR wavelet functions on level } j$$

$$\mathcal{H}_J = \Phi_J \cup \bigcup_{j\geq J} \Psi_j : \text{HAAR functions on all levels} \geq J$$

$$\Psi = \mathcal{H} = \{\psi_{j,k}\}_{j,k\in\mathbb{Z}} : \text{ HAAR wavelet functions (all levels)}$$

Localization (1)

- The Fourier transform of $\phi(t)$ is

$$\widehat{\phi}(s) = \frac{\sin(\pi s)}{\pi s} \cdot e^{-i\pi s}$$

- The function $|\widehat{\phi}(s)|$
  - has its maximum at $s = 0$
  - its first positive root at $s = \pm 1$
  - decreases as $1/s$
- The Fourier transform of $\psi(t)$ is

$$\widehat{\psi}(s) = \frac{i(1 - \cos(\pi s))}{\pi s} \cdot e^{-i\pi s} = \frac{2i}{\pi s} \sin^2(\pi s/2) \cdot e^{-i\pi s}$$

- The function $|\widehat{\psi}(s)|$
  - has its first maximum at $s_0 \approx 0.7420192\ldots$
  - its first positive root at $s = 2$
  - decreases as $1/s$

Localization (2)

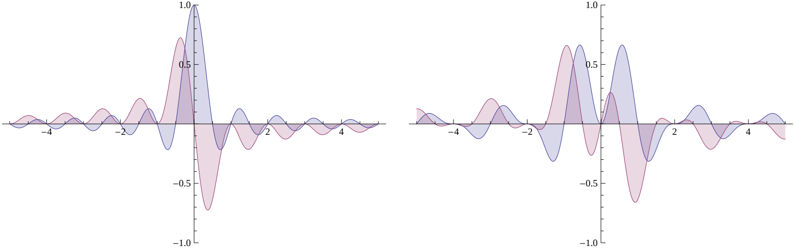


Figure: Real and imaginary parts of $\widehat{\phi}(s)$ and $\widehat{\psi}(s)$

Localization (3)

- One may state:
  - ϕ and ψ are well localized in the time/space domain at $s = 0$
  - ϕ and ψ are quite well localized in the frequency domain (but not too well, because $\widehat{\phi}$ and $\widehat{\psi}$ have infinite variance)
- In sharp contrast to Fourier analysis one has reasonably good localization both in the time/space domain and in the frequency domain

Normalization

- Normalization of the scaling functions

$$\int_{\mathbb{R}} \phi_{j,k} = 2^{-j/2} \qquad\qquad \|\phi_{j,k}\|_2^2 = \int_{\mathbb{R}} |\phi_{j,k}|^2 = 1$$

- Normalization of the wavelet functions

$$\int_{\mathbb{R}} \psi_{j,k} = 0 \qquad\qquad \|\psi_{j,k}\|_2^2 = \int_{\mathbb{R}} |\psi_{j,k}|^2 = 1$$

Orthogonality

- For all $i, j, k, \ell \in \mathbb{Z}$:

$$\langle \phi_{j,k} \,|\, \phi_{j,\ell} \rangle = \int_{\mathbb{R}} \phi_{j,k}\, \phi_{j,\ell} = \delta_{k,\ell}$$

$$\langle \psi_{i,k} \,|\, \psi_{j,\ell} \rangle = \int_{\mathbb{R}} \psi_{i,k}\, \psi_{j,\ell} = \delta_{i,j}\delta_{k,\ell}$$

$$\langle \phi_{i,k} \,|\, \psi_{j,\ell} \rangle = \int_{\mathbb{R}} \phi_{i,k}\, \psi_{j,\ell} = 0 \quad \text{if } j \geq i$$

- The following families are orthogonal families of functions in $\mathcal{L}^2(\mathbb{R})$:
  1. The HAAR scaling familiy Φ_j on a fixed level j ($j \in \mathbb{Z}$)
  2. The HAAR wavelet family $\Psi = \mathcal{H}$
  3. The HAAR family $\mathcal{H}_J$ for fixed $J \in \mathbb{Z}$.
- Warning: scaling functions $\phi_{j,k}$ and $\phi_{\ell,m}$ belonging to different resolutions, i.e., $j \neq \ell$, are not orthogonal in general.
  $\Phi = \bigcup_j \Phi_j$ is not an orthogonal family!

Scaling and wavelets (1)

- Fundamental relation between HAAR scaling functions and HAAR wavelet functions:

$$\phi(t) = \phi(2t) + \phi(2t-1) = \frac{1}{\sqrt{2}}(\phi_{1,0}(t) + \phi_{1,1}(t)) \quad \text{scaling eqn}$$

$$\psi(t) = \phi(2t) - \phi(2t-1) = \frac{1}{\sqrt{2}}(\phi_{1,0}(t) - \phi_{1,1}(t)) \quad \text{wavelet eqn}$$

- Matrix version:

$$\begin{bmatrix} \phi_{0,0}(t) \\ \psi_{0,0}(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \phi_{1,0}(t) \\ \phi_{1,1}(t) \end{bmatrix}$$

- The transformation matrix (HADAMARD-matrix)

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

This matrix is orthogonal, i.e., $H^{-1} = H^t (= H)$

Scaling and wavelets (2)

- Consequently

$$\begin{bmatrix} \phi_{1,0}(t) \\ \phi_{1,1}(t) \end{bmatrix} = H \cdot \begin{bmatrix} \phi_{0,0}(t) \\ \psi_{0,0}(t) \end{bmatrix}$$

- By dilation and translation one obtains for all $j, k \in \mathbb{Z}$:

$$\begin{bmatrix} \phi_{j,k}(t) \\ \psi_{j,k}(t) \end{bmatrix} = H \cdot \begin{bmatrix} \phi_{j+1,2k}(t) \\ \phi_{j+1,2k+1}(t) \end{bmatrix}$$

$$\begin{bmatrix} \phi_{j+1,2k}(t) \\ \phi_{j+1,2k+1}(t) \end{bmatrix} = H \cdot \begin{bmatrix} \phi_{j,k}(t) \\ \psi_{j,k}(t) \end{bmatrix}$$

Vector spaces (1)

- Approximation spaces ($j \in \mathbb{Z}$)

$$\mathcal{V}_j = \overline{span}\,\Phi_j = \{\text{dyadic } \mathcal{L}^2\text{-step functions with step width } 2^{-j}\}$$

- Detail spaces ($j \in \mathbb{Z}$)

$$\mathcal{W}_j = \overline{span}\,\Psi_j = \{\text{balanced dyadic } \mathcal{L}^2\text{-step fns. with step width } 2^{-(j+1)}\}$$

- For $j \in \mathbb{Z}$ both

$$\Phi_{j+1} = \{\phi_{j+1,k}\}_{k\in\mathbb{Z}} \quad \text{and} \quad \Phi_j \cup \Psi_j = \{\phi_{j,k}, \psi_{j,k}\}_{k\in\mathbb{Z}}$$

  are orthonormal bases of $\mathcal{V}_{j+1}$

- H is (essentially) the matrix of a basis change between Φ_{j+1} and $\Phi_j \cup \Psi_j$

Vector spaces (2)

- For all $j \in \mathbb{N}$ one has

$$\mathcal{V}_j \subset \mathcal{V}_{j+1}, \quad \mathcal{W}_j \subset \mathcal{V}_{j+1}$$

and even

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$$

- Consequently

$$\{0\} \subset \cdots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{L}^2(\mathbb{R})$$

- For all $j \geq k$ one has

$$\mathcal{V}_{j+1} = \mathcal{V}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \cdots \oplus \mathcal{W}_{j-1} \oplus \mathcal{W}_j$$

- The vector space $\mathcal{V}_{j+1}$ has as a basis the family Φ_{j+1} and also, for each $k \leq j$, the family

$$\Phi_k \cup \Psi_k \cup \Psi_{k+1} \cup \cdots \cup \Psi_{j-1} \cup \Psi_j$$

Vector spaces (3)

- The relation $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$ says that the vector space has two bases:
  - the basis $\Phi_{j+1} = \{\phi_{j+1,k}\}_{k\in\mathbb{Z}}$
  - and the basis $\Phi_j \cup \Psi_j = \{\phi_{j,k}\}_{k\in\mathbb{Z}} \cup \{\psi_{j,k}\}_{k\in\mathbb{Z}}$.
- For each f in $\mathcal{V}_{j+1}$ there exist $g \in \mathcal{V}_j$ and $h \in \mathcal{W}_j$ such that

$$f = g + h$$

  g and h are orthogonal and they are uniquely determined
- The mapping
  - $f \mapsto (g, h)$ is called *analysis mapping*, as it decomposes f in an "approximating" ("low-frequency") part g and a "detailing" ("high-frequency") part h
  - $(g, h) \mapsto f$ is called *synthesis mapping*, as it reconstructs f from its low- and high-frequency parts

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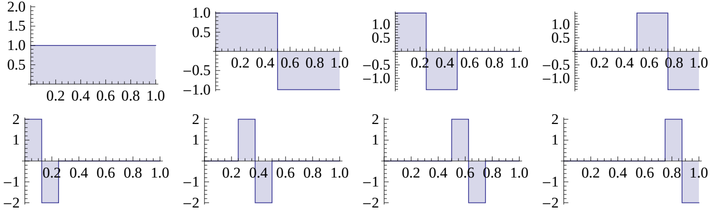

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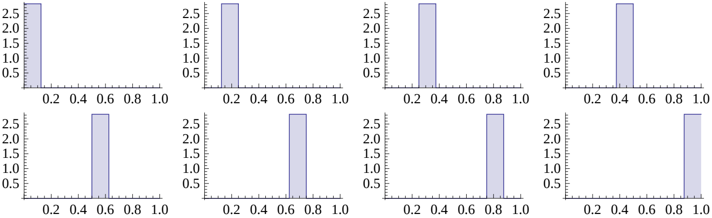


Figure: Two bases of $\mathcal{V}_3$ (restricted to $[0, 1)$)

Scaling and wavelet coefficients (1)

- Inner product in $\mathcal{L}^2(\mathbb{R})$

$$\langle f \mid g \rangle = \int_{\mathbb{R}} f(t)\,\overline{g(t)}\,dt$$

- For $f \in \mathcal{L}^2(\mathbb{R})$ and $j, k \in \mathbb{Z}$
  - the HAAR *scaling coefficients* (or *approximation coeffs*) of f are the

$$a_{j,k} = \langle f \mid \phi_{j,k} \rangle = 2^{j/2} \int_{I_{j,k}} f(t)\,dt$$

  - the HAAR *wavelet coefficients* (or *detail coeffs*) of f are the

$$d_{j,k} = \langle f \mid \psi_{j,k} \rangle = 2^{j/2} \left(\int_{I_{j+1,2k}} f(t)\,dt - \int_{I_{j+1,2k+1}} f(t)\,dt \right)$$

Scaling and wavelet coefficients (2)

- The coefficients $a_{j,k} = \langle f \mid \phi_{j,k} \rangle$ and $d_{j,k} = \langle f \mid \psi_{j,k} \rangle$ only depend on the behavior of f on the dyadic intervall $I_{j,k}$!
- $a_{j,k} = \langle f \mid \phi_{j,k} \rangle$ means "averaging" or "smoothing" and is called *approximation coefficient* of f
- $d_{j,k} = \langle f \mid \psi_{j,k} \rangle$ records the variation of f between the left and the right subintervals of $I_{j,k}$ and is called *detail coefficient* of f, as it emphasizes changes (fluctuations)

Scaling- and wavelet coefficients (3)

- From the basic relation $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$ one has immediately a recursion for the scaling and for the wavelet coefficients:
- *Analysis*: For all $j, k \in \mathbb{Z}$ one has

$$\begin{bmatrix} a_{j,k} \\ d_{j,k} \end{bmatrix} = H \cdot \begin{bmatrix} a_{j+1,2k} \\ a_{j+1,2k+1} \end{bmatrix}$$

- *Synthesis*: For all $j, k \in \mathbb{Z}$ one has

$$\begin{bmatrix} a_{j+1,2k} \\ a_{j+1,2k+1} \end{bmatrix} = H \cdot \begin{bmatrix} a_{j,k} \\ d_{j,k} \end{bmatrix}$$

- Equivalently:

$$a_{j+1,2k} \cdot \phi_{j+1,2k} + a_{j+1,2k+1} \cdot \phi_{j+1,2k+1} = a_{j,k} \cdot \phi_{j,k} + d_{j,k} \cdot \psi_{j,k}$$

and

$$\sum_k a_{j+1,k}\phi_{j+1,k} = \sum_\ell a_{j,\ell}\phi_{j,\ell} + \sum_m d_{j,m}\psi_{j,m}$$

Projection operators

- The last identity says that for any $f \in \mathcal{L}(\mathbb{R})$

$$P_{j+1}f = P_j f + Q_j f,$$

- where P_j and Q_j are the orthogonal projections of functions $f \in \mathcal{L}^2(\mathbb{R})$ onto the subspaces $\mathcal{V}_j$ and $\mathcal{W}_j$:

$$P_j : \mathcal{L}^2(\mathbb{R}) \to \mathcal{V}_j : f \mapsto \sum_{k \in \mathbb{Z}} \langle\, f \,|\, \phi_{j,k} \,\rangle\, \phi_{j,k}$$

$$Q_j : \mathcal{L}^2(\mathbb{R}) \to \mathcal{W}_j : f \mapsto \sum_{k \in \mathbb{Z}} \langle\, f \,|\, \psi_{j,k} \,\rangle\, \psi_{j,k}$$

- These projections provide the optimal approximations w.r.t. the $\mathcal{L}^2$-norm of f within the spaces $\mathcal{V}_j$ and $\mathcal{W}_j$
- These linear projection operators satisfy

$$P_{j+1} = P_j + Q_j \qquad (j \in \mathbb{N})$$

HAAR families on [0, 1] (1)

- For $J \geq 0$ the family

$$\mathcal{H}_J = \{\phi_{J,k}\}_{0 \leq k < 2^j} \cup \{\psi_{j,k}\}_{j \geq J, 0 \leq k < 2^J}$$

is the family of HAAR functions of level J on the interval $[0, 1]$
- In this case the vector spaces $\mathcal{V}_j$ and $\mathcal{W}_j$ have finite dimension:

$$\dim \mathcal{V}_j = \dim \mathcal{W}_j = 2^j \quad (j \geq 0)$$

- Similarly for arbitrary finite intervals of $\mathbb{R}$

HAAR families on [0, 1] (2)

- For each $J \geq 0$ the family $\mathcal{H}_J$ is a complete ONS (Hilbert basis) for $\mathcal{L}^2[0,1]$
- Idea of proof:
  - The *continuos* functions are dense in $\mathcal{L}^2[0,1]$
  - Every continuous function on a finite interval can be approxiamted arbirarily well w.r.t. the $\mathcal{L}^2$-norm by dyadic step-functions
  - Every dyadic step function with step width 2^{-j} belongs to $\mathcal{V}_j$ and can be represented in each of the bases $\mathcal{H}_J$ $(J \geq 0)$

HAAR families on $\mathbb{R}$ (1)

- Now: $\mathcal{H}_J$ with $J \in \mathbb{Z}$ denotes the HAAR family for $\mathbb{R}$
- Recall:
  $\mathcal{V}_j$ and $\mathcal{W}_j$ are $\mathcal{L}^2$-closures of the vector spaces spanned by HAAR functions Φ_j and Ψ_j within $\mathcal{L}^2(\mathbb{R})$:

  $$\mathcal{V}_j = \overline{span}\,\{\phi_{j,k}\}_{k\in\mathbb{Z}}\,, \quad \mathcal{W}_j = \overline{span}\,\{\psi_{j,k}\}_{k\in\mathbb{Z}}$$

  Infinite sums are legitimate, but they must converge in the $\mathcal{L}^2$ sense
- *Approximation* and *detail* as projection operators:

$$P_j : \mathcal{L}^2(\mathbb{R}) \to \mathcal{V}_j : f \mapsto \sum_{k\in\mathbb{Z}} \langle\, f \mid \phi_{j,k}\,\rangle\, \phi_{j,k},$$

$$Q_j : \mathcal{L}^2(\mathbb{R}) \to \mathcal{W}_j : f \mapsto \sum_{k\in\mathbb{Z}} \langle\, f \mid \psi_{j,k}\,\rangle\, \psi_{j,k}$$

HAAR families on $\mathbb{R}$ (2)

- Operators P_j and Q_j are linear transformations
- Operators P_j and Q_j are projections, i.e., the satisfy $P_j^2 = P_j, Q_j^2 = Q_j$
- For $k \geq j$ one has $P_k|_{\mathcal{V}_j} = id$
- For $k \neq j$ one has $Q_k|_{\mathcal{W}_j} = 0$
- $||P_j f||_2 \leq ||f||_2$ und $||Q_j f||_2 \leq ||f||_2$
- $Q_j = P_{j+1} - P_j$
- For $f \in \mathcal{C}_c^0(\mathbb{R})$ (i.e., continuous with compact support) one has convergence (w.r.t. $\mathcal{L}^2$) $\quad P_j f \rightarrow_{\infty} f$ and $P_j f \rightarrow_{-\infty} 0$
- For $f \in \mathcal{L}^2(\mathbb{R})$ operators $P_j f$ and $Q_j f$ are defined by approximating arbitrary functions by continuous functions with compact support

HAAR families on $\mathbb{R}$ (3)

- Scheme of *multiresolution analysis* (MRA)
  - Nesting
  $$\cdots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots$$
  - Completeness
  $$\lim_{j \to \infty} \mathcal{V}_j = \bigcup_{j \in \mathbb{Z}} \mathcal{V}_j = \mathcal{L}^2(\mathbb{R})$$
  - Separation
  $$\bigcap_{j \in \mathbb{Z}} \mathcal{V}_j = \{0\}$$
  - Scaling
  $$f \in \mathcal{V}_0 \quad \Leftrightarrow \quad D_{2^j} f \in \mathcal{V}_j \quad (f \in \mathcal{L}^2(\mathbb{R}), j \in \mathbb{Z})$$
  - Translation and orthogonality
  $$\overline{span}\{T_k \phi\}_{k \in \mathbb{Z}} = \overline{span}\{\phi(t-k)\}_{k \in \mathbb{Z}} = \mathcal{V}_0$$

HAAR families on $\mathbb{R}$ (4)

- Theorem:
  For each $J \in \mathbb{Z}$ the family $\mathcal{H}_J$ is a complete ONS (i.e., a Hilbert basis) for full signal space $\mathcal{L}^2(\mathbb{R})$
  - Idea of proof
  - continuous functions with compact support are dense in $\mathcal{L}^2(\mathbb{R})$. It suffices therefore to refer to the situation of finite intervals
  - properties of the projections P_j, Q_j etc. carry over from finite to infinite intervals in a similar fashion
- Theorem:
  The HAAR family of all balanced dyadic step functions

$$\mathcal{H} = \Psi = \{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$$

is a complete ONS (i.e., a Hilbert basis) for the full signal space $\mathcal{L}^2(\mathbb{R})$

Notation (1)

- $A \otimes B$: Tensor product ("Kronecker product") of matrices

  If $A = [a_{i,j}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ and $B = [b_{k,\ell}]_{\substack{1 \le k \le p \\ 1 \le \ell \le q}}$ then the $(m \cdot p) \times (n \cdot q)$ matrix $A \otimes B$ is defined by

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \dots & a_{m,n}B \end{bmatrix}$$

- Example

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}$$

Notation(2)

- I_n is the $(n \times n)$ unit matrix
  0_n is the $(n \times n)$ zero matrix
- $A^\dagger$: adjoint matrix of A (transpose and complex-conjugate)
- the Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Example

$$H \otimes H = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Haar approximation and detail matrices

- For $n \geq 1$ define

$$A_n = I_n \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad D_n = I_n \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These are matrices of format $(2n \times n)$

$$A_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & \\ 1 & & & \\ & 1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 1 \end{bmatrix} \quad D_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ & & \ddots & \\ & & & 1 \\ & & & -1 \end{bmatrix}$$

Properties of HAAR matrices

- Orthogonality

$$A_n^\dagger \cdot A_n = I_n \qquad A_n \cdot A_n^\dagger = \frac{1}{2}\, I_n \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$D_n^\dagger \cdot D_n = I_n \qquad D_n \cdot D_n^\dagger = \frac{1}{2}\, I_n \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A_n^\dagger \cdot D_n = 0_n \qquad A_n \cdot D_n^\dagger = \frac{1}{2}\, I_n \otimes \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$D_n^\dagger \cdot A_n = 0_n \qquad D_n \cdot A_n^\dagger = \frac{1}{2}\, I_n \otimes \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

- $\begin{bmatrix} A_n & D_n \end{bmatrix}$ is an orthogonal matrix, i.e.,

$$\begin{bmatrix} A_n & D_n \end{bmatrix} \begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix} = I_{2n} = \begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix} \begin{bmatrix} A_n & D_n \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} A_n & D_n \end{bmatrix} \cdot \begin{bmatrix} A_n & D_n \end{bmatrix}^\dagger &= \begin{bmatrix} A_n & D_n \end{bmatrix} \begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix} \\
&= A_n \cdot A_n^\dagger + D_n \cdot D_n^\dagger \\
&= \frac{1}{2} I_n \otimes \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \\
&= \frac{1}{2} I_n \otimes \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I_{2n}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} A_n & D_n \end{bmatrix}^\dagger \cdot \begin{bmatrix} A_n & D_n \end{bmatrix} &= \begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix} \begin{bmatrix} A_n & D_n \end{bmatrix} \\
&= \begin{bmatrix} A_n^\dagger \cdot A_n & A_n^\dagger \cdot D_n \\ D_n^\dagger \cdot A_n & D_n^\dagger \cdot D_n \end{bmatrix} \\
&= \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix} = I_{2n}
\end{aligned}$$

Haar transform (1)

- The matrix $\begin{bmatrix} A_n & D_n \end{bmatrix}$ is the matrix of a one-level discrete HAAR transform of signals (vectors) of length $2n$
- For $\mathbf{a}_{2n}$ a (row) vector of length $2n$ let

$$\mathbf{a}_{2n} \mapsto \mathbf{a}_{2n} \cdot \begin{bmatrix} A_n & D_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}'_n & \mathbf{d}'_n \end{bmatrix}$$

  This is a linear transformation of the vector space $\mathbb{C}^{2n}$. One has

$$\mathbf{a}'_n = \mathbf{a}_{2n} \cdot A_n, \quad \mathbf{d}'_n = \mathbf{a}_{2n} \cdot D_n$$

- Since this is an orthogonal transformation, one can simply revert this relation:

$$\begin{aligned} \mathbf{a}_{2n} &= \begin{bmatrix} \mathbf{a}'_n & \mathbf{d}'_n \end{bmatrix} \cdot \begin{bmatrix} A_n & D_n \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{a}'_n & \mathbf{d}'_n \end{bmatrix} \cdot \begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix} \\ &= \mathbf{a}'_n \cdot A_n^\dagger + \mathbf{d}'_n \cdot D_n^\dagger \end{aligned}$$

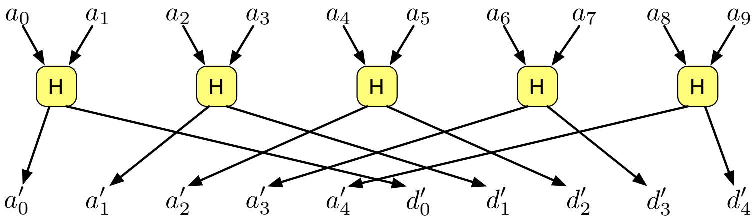


Figure: One-level Haar transform (analysis) of a vector of length 10

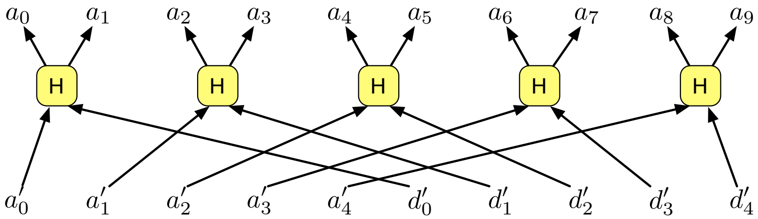


Figure: One-level Haar transform (synthesis) of a vector of length 10

Inductive definition of multilevel HAAR transform

- For $k = 1$ one has

$$\mathtt{DHT}_1 : \mathbb{C}^{2n} \to \mathbb{C}^{2n} : \mathbf{a}_{2n} \mapsto \mathbf{a}_{2n} \begin{bmatrix} A_n & D_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}'_n & \mathbf{d}'_n \end{bmatrix}$$

- Assume that $\mathtt{DHT}_k : \mathbb{C}^{2^k n} \to \mathbb{C}^{2^k n}$ has been defined, then $\mathtt{DHT}_{k+1} : \mathbb{C}^{2^{k+1} n} \to \mathbb{C}^{2^{k+1} n}$ is defined by

$$\mathbf{a}_{2^{k+1} n} \mapsto \begin{bmatrix} \mathtt{DHT}_k(\mathbf{a}'_{2^k n}) & \mathbf{d}'_{2^k n} \end{bmatrix}$$

  where $\mathtt{DHT}_1(\mathbf{a}_{2^{k+1} n}) = \begin{bmatrix} \mathbf{a}'_{2^k n} & \mathbf{d}'_{2^k n} \end{bmatrix}$
- By induction it follows that the $\mathtt{DHT}_k$ are orthogonal transformations
- The inverse transformations are obtained by inverting the one-level transformations as indicated above

Multilevel HAAR transform

- One may write in a suggestive manner

$$\mathtt{DHT}_k(\mathbf{a}_{2^k n}) = \left[\mathbf{a}_n^{(k)} \quad \mathbf{d}_n^{(k)} \quad \mathbf{d}_{2n}^{(k-1)} \quad \mathbf{d}_{4n}^{(k-2)} \quad \ldots \quad \mathbf{d}''_{2^{k-2}n} \quad \mathbf{d}'_{2^{k-1}n}\right],$$

- This can be read in the light of the basis decomposition

$$\mathcal{V}_{J+k} = \mathcal{V}_J \oplus \mathcal{W}_J \oplus \mathcal{W}_{J+1} \oplus \mathcal{W}_{J+2} \oplus \cdots \oplus \mathcal{W}_{J+k-1}$$

  as follows: if the entries of $\mathbf{a}_{2^k n}$ are the coefficients of a function f w.r.t. the basis $\Phi_{J+k} = \{\phi_{J+k,m}\}$, then the entries of the vector $\mathtt{DHT}_k(\mathbf{a}_{2^k n})$ are the coefficients of f w.r.t. the bases
  - $\Phi_J = \{\phi_{J,m}\}$ in $\mathcal{V}_J$ $\qquad a^{(k)}_{J+k,m} = \langle f \mid \phi_{J+k,m}\rangle$,
  - $\Psi_J = \{\psi_{J,m}\}$ in $\mathcal{W}_J$ $\qquad d^{(k)}_{J+k,m} = \langle f \mid \psi_{J,J+k}\rangle$,
  - $\Psi_{J+1} = \{\psi_{J+1,m}\}$ in $\mathcal{W}_{J+1}$ $\qquad d^{(k-1)}_{J+k-1,m} = \langle f \mid \phi_{J+k-1,m}\rangle$,
  - $\Psi_{J+2} = \{\psi_{J+2,m}\}$ in $\mathcal{W}_{J+2}$ $\qquad a^{(k-2)}_{J+k-2,m} = \langle f \mid \phi_{J+k-2,m}\rangle$,
  - $\ldots$ $\qquad \ldots$
  - $\Psi_{J+k-1} = \{\psi_{J+k-1,m}\}$ in $\mathcal{W}_{J+k-1}$ $\qquad d'_{J+1,m} = \langle f \mid \phi_{J+1,m}\rangle$.

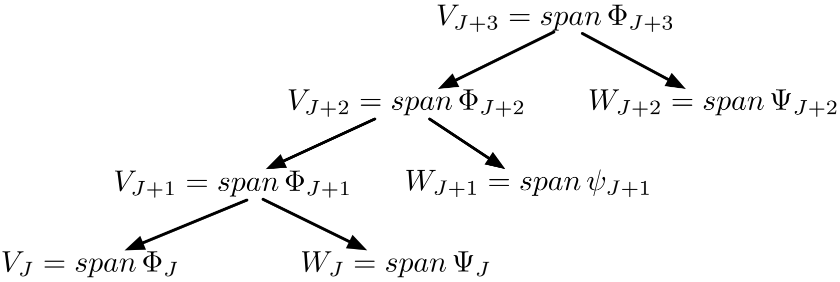


Figure: Scheme of a three-level Haar transform (analysis)

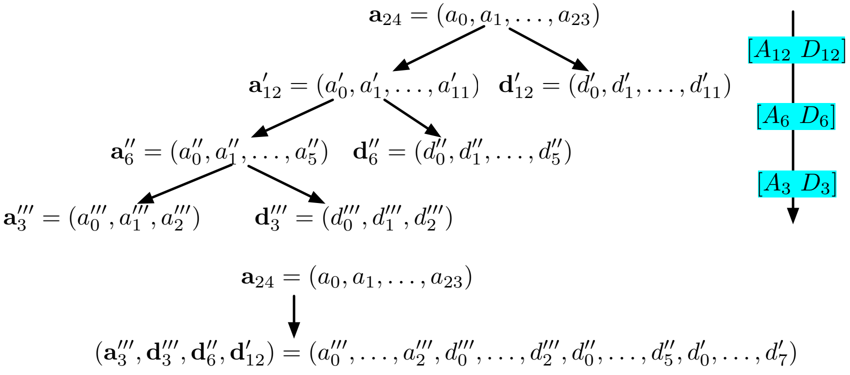


Figure: Three-level Haar transform (analysis) of a vector of length 24

$$\mathbf{a}_{24} = (a_0, a_1, \ldots, a_{23})$$

$$[A_{12}\ D_{12}]^\dagger$$

$$\mathbf{a}'_{12} = (a'_0, a'_1, \ldots, a'_{11}) \quad \mathbf{d}'_{12} = (d'_0, d'_1, \ldots, d'_{11})$$

$$[A_6\ D_6]^\dagger$$

$$\mathbf{a}''_6 = (a''_0, a''_1, \ldots, a''_5) \quad \mathbf{d}''_6 = (d''_0, d''_1, \ldots, d''_5)$$

$$[A_3\ D_3]^\dagger$$

$$\mathbf{a}'''_3 = (a'''_0, a'''_1, a'''_2) \qquad \mathbf{d}'''_3 = (d'''_0, d'''_1, d'''_2)$$

$$\mathbf{a}_{24} = (a_0, a_1, \ldots, a_{23})$$

$$\uparrow$$

$$(\mathbf{a}'''_3, \mathbf{d}'''_3, \mathbf{d}''_6, \mathbf{d}'_{12}) = (a'''_0, \ldots, a'''_2, d'''_0, \ldots, d'''_2, d''_0, \ldots, d''_5, d'_0, \ldots, d'_7)$$

Figure: Three-level Haar transform (synthesis) of a vector of length 24

Complexity of the HAAR transform

- In practice: the multiplication of a vector of length $2n$ with the matrix $[A_n \, D_n]$ should NEVER be implemented as a vector$\times$matrix operation, because these matrices a very sparse.
  One needs only *const* $\times\, 2n$ elementary operations
- Computation of $\mathtt{DHT}_k$ on a vector of length $2^k n$ needs then only

$$const \cdot \left(2^k + 2^{k-1} + \cdots + 2^1\right) n = \mathcal{O}(2^k n)$$

  elementary operations, which is *linear* (!) in the input size
- The same argument holds for the complexity of the inverse transform $\mathtt{DHT}_k^{-1}$

HAAR transform as filtering operation

- HAAR wavelet analysis and HAAR wavelet synthesis can be understood as filtering operations
  - The A-matrices act as low-pass filters
  - the D-matrices act as high-pass filters
- To make this precise, it is convenient to consider bi-infinite sequences of (complex) values ("signals") as inputs

$$\mathbf{a} = (a[n])_{n \in \mathbb{Z}} = (\ldots a[-2], a[-1], a[0], a[1], a[2], \ldots)$$

Low-pass filter

- approximation (low-pass) matrix
  $\mathcal{A} = [a_{i,j}]_{i,j \in \mathbb{Z}}$ is a matrix of infinite size with

$$a_{i,j} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } i = j \text{ or } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

visualized by

$$\mathcal{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & & & & & \\ \ddots & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & 1 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}$$

High-pass filter

- Detail (high pass) matrix
  $\mathcal{D} = [d_{i,j}]_{i,j\in\mathbb{Z}}$ is an infinite matrix with

$$d_{i,j} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } i = j \\ -\frac{1}{\sqrt{2}} & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

  visualized by

$$\mathcal{D} = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & & & & & \\ \ddots & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}$$

Adjoint low-pass filter

- Adjoint approximation matrix $\mathcal{A}^\dagger = \left[a^\dagger_{i,j}\right]_{i,j\in\mathbb{Z}}$ with

$$a^\dagger_{i,j} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } i = j \text{ or } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

visualized by

$$\mathcal{A}^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & \ddots & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & 1 & 1 & \\ & & & & 1 & \ddots \\ & & & & & \ddots \end{bmatrix}$$

Adjoint high-pass filter

- Adjoint detail matrix $\mathcal{D}^\dagger = \left[d^\dagger_{i,j} \right]_{i,j \in \mathbb{Z}}$

$$d^\dagger_{i,j} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } i = j \\ -\frac{1}{\sqrt{2}} & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

visualized by

$$\mathcal{D}^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} \ddots & \ddots & & & & \\ & 1 & -1 & & & \\ & & 1 & -1 & & \\ & & & 1 & -1 & \\ & & & & 1 & \ddots \\ & & & & & \ddots \end{bmatrix}$$

Filtering operations as convolution (1)

- Acting with these matrices from the right (!) on a signal vector $\mathbf{a} = (\ldots a[-1], a[0], a[1], a[2] \ldots) = (a[n])_{n\in\mathbb{Z}}$ gives:

$$\mathbf{a} \cdot \mathcal{A} = \left(\frac{a[n] + a[n+1]}{\sqrt{2}} \right)_{n\in\mathbb{Z}}$$

$$\mathbf{a} \cdot \mathcal{D} = \left(\frac{a[n] - a[n+1]}{\sqrt{2}} \right)_{n\in\mathbb{Z}}$$

$$\mathbf{a} \cdot \mathcal{A}^\dagger = \left(\frac{a[n] + a[n-1]}{\sqrt{2}} \right)_{n\in\mathbb{Z}}$$

$$\mathbf{a} \cdot \mathcal{D}^\dagger = \left(\frac{a[n] - a[n-1]}{\sqrt{2}} \right)_{n\in\mathbb{Z}}$$

Filtering operations as convolution (2)

Defining HAAR filters as

$$h_\phi[n] = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_\psi[n] = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0 \\ -\frac{1}{\sqrt{2}} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

the matrix multiplications turn out to be convolution operations:

$$\mathbf{a} \mapsto \mathbf{a} \cdot \mathcal{A} = \mathbf{a} \star (h_\phi[-n])_{n \in \mathbb{Z}} \quad \text{convolution with } (h_\phi[-n])_{n \in \mathbb{Z}}$$

$$\mathbf{a} \mapsto \mathbf{a} \cdot \mathcal{D} = \mathbf{a} \star (h_\psi[-n])_{n \in \mathbb{Z}} \quad \text{convolution with } (h_\psi[-n])_{n \in \mathbb{Z}}$$

$$\mathbf{a} \mapsto \mathbf{a} \cdot \mathcal{A}^\dagger = \mathbf{a} \star (h_\phi[n])_{n \in \mathbb{Z}} \quad \text{convolution with } (h_\phi[n])_{n \in \mathbb{Z}}$$

$$\mathbf{a} \mapsto \mathbf{a} \cdot \mathcal{D}^\dagger = \mathbf{a} \star (h_\psi[n])_{n \in \mathbb{Z}} \quad \text{convolution with } (h_\psi[n])_{n \in \mathbb{Z}}$$

Downsampling and upsampling

downsampling $\downarrow_2$ and *upsampling* $\uparrow_2$

$$\mathbf{a}\downarrow_2 = (a[2n])_{n\in\mathbb{Z}} \quad = (\ldots a[-2], a[0], a[2], a[4]\ldots)$$
$$\mathbf{a}\uparrow_2 = (a[n/2]\,\mathbf{1}_{even}(n))_{n\in\mathbb{Z}} \quad = (\ldots a[-1], 0, a[0], 0, a[1], 0, a[2], 0\ldots)$$

Written in matrix form:

$$\downarrow_2 \;=\; \begin{bmatrix} \ddots & & & \\ \ddots & 0 & & \\ & 1 & & \\ & 0 & 0 & & \\ & & 1 & & \\ & & 0 & 0 & \\ & & & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix}, \quad \uparrow_2 \;=\; \begin{bmatrix} \ddots & \ddots & & & & & & \\ & 0 & 1 & 0 & & & & \\ & & & 0 & 1 & 0 & & \\ & & & & & 0 & 1 & 0 \\ & & & & & & \ddots & \ddots \end{bmatrix}$$

The matrix for upsampling $\uparrow_2$ is the adjoint (transpose) of the downsampling

HAAR wavelet transform as a filtering operation

In perfect analogy to the HAAR wavelet transform one has the transforms for analysis

- low-pass filtering followed by downsampling:

$$\mathcal{A} \circ \downarrow_2 \quad : \; \mathbf{a} \mapsto (\mathbf{a}\,\mathcal{A}) \downarrow_2 = \left(\frac{a[2n] + a[2n+1]}{\sqrt{2}} \right)_{n \in \mathbb{Z}} = \mathbf{a}'$$

- high-pass filtering followed by downsampling:

$$\mathcal{D} \circ \downarrow_2 \quad : \; \mathbf{a} \mapsto (\mathbf{a}\,\mathcal{D}) \downarrow_2 = \left(\frac{a[2n] - a[2n+1]}{\sqrt{2}} \right)_{n \in \mathbb{Z}} = \mathbf{d}'$$

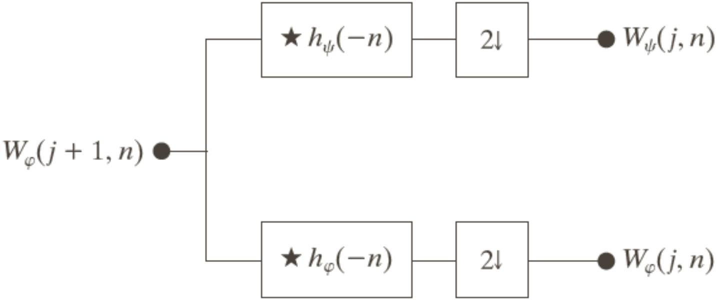


Figure: HAAR analysis (1 level)

Wavelet reconstruction as a filtering operation (1)

- The synthesis transformation for reconstruction of **a** uses upsampling and the adjoint matrices:

$$\mathbf{a}' \uparrow_2 \mathcal{A}^\dagger + \mathbf{d}' \uparrow_2 \mathcal{D}^\dagger = \mathbf{a}$$

- Check how the operations $\mathbf{a} \mapsto \mathbf{a} \uparrow_2 \mathcal{A}^\dagger$ and $\mathbf{a} \mapsto \mathbf{a} \uparrow_2 \mathcal{D}^\dagger$ act on an arbitrary sequence $\mathbf{a} = (a[n])_{n\in\mathbb{Z}}$:

$$\mathbf{a} \uparrow_2 \mathcal{A}^\dagger = \left(\frac{a[\lfloor n/2 \rfloor]}{\sqrt{2}}\right)_{n\in\mathbb{Z}}$$

$$\mathbf{a} \uparrow_2 \mathcal{D}^\dagger = \left(\frac{(-1)^n a[\lfloor n/2 \rfloor]}{\sqrt{2}}\right)_{n\in\mathbb{Z}}$$

Wavelet reconstruction as a filtering operation (2)

- We have

$$\mathbf{a}' \uparrow_2 \mathcal{A}^\dagger + \mathbf{d}' \uparrow_2 \mathcal{D}^\dagger = \left(\frac{a'[\lfloor n/2 \rfloor]}{\sqrt{2}}\right)_{n\in\mathbb{Z}} + \left(\frac{(-1)^n d'[\lfloor n/2 \rfloor]}{\sqrt{2}}\right)_{n\in\mathbb{Z}}$$
$$= \left(\frac{a'[\lfloor n/2 \rfloor] + (-1)^n d'[\lfloor n/2 \rfloor]}{\sqrt{2}}\right)_{n\in\mathbb{Z}}$$

- so that for n even:

$$\frac{1}{\sqrt{2}}\left(a'[\lfloor n/2 \rfloor] + (-1)^n d'[\lfloor n/2 \rfloor]\right) = \frac{1}{\sqrt{2}}\left(\frac{a[n] + a[n+1]}{\sqrt{2}} + \frac{a[n] - a[n+1]}{\sqrt{2}}\right) = a[n]$$

- and for n odd:

$$\frac{1}{\sqrt{2}}\left(a'[\lfloor n/2 \rfloor] + (-1)^n d'[\lfloor n/2 \rfloor]\right) = \frac{1}{\sqrt{2}}\left(\frac{a[n-1] + a[n]}{\sqrt{2}} - \frac{a[n-1] - a[n]}{\sqrt{2}}\right) = a[n]$$

Wavelet reconstruction as a filtering operation (3)

- Putting things together:

$$\mathcal{A} \downarrow_2 \uparrow_2 \mathcal{A}^\dagger + \mathcal{D} \downarrow_2 \uparrow_2 \mathcal{D}^\dagger = Id$$

- Using the fact that downsampling and upsampling are adjoint operations, one can write this in a more concise way as: for $A = \mathcal{A} \downarrow_2, D = \mathcal{D} \downarrow_2$ one has

$$A\,A^\dagger + D\,D^\dagger = Id.$$

- The following relations between the transformations are easily checked:

$$A^\dagger\,A = Id, \quad D^\dagger\,D = Id, \quad A^\dagger\,D = 0 = D^\dagger\,A.$$

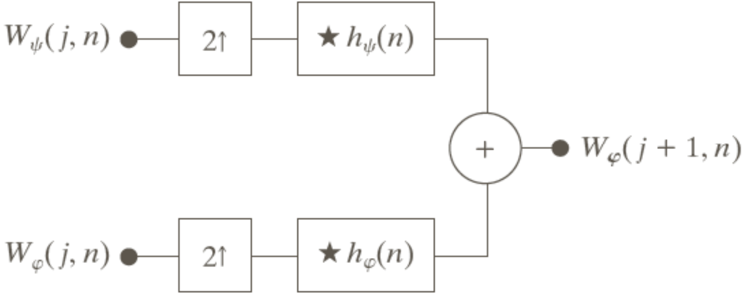


Figure: HAAR synthesis (one level)

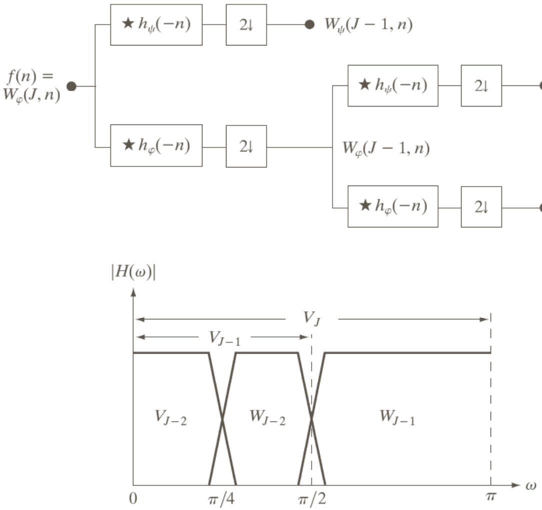


Figure: HAAR analysis (2 levels) and frequency separation

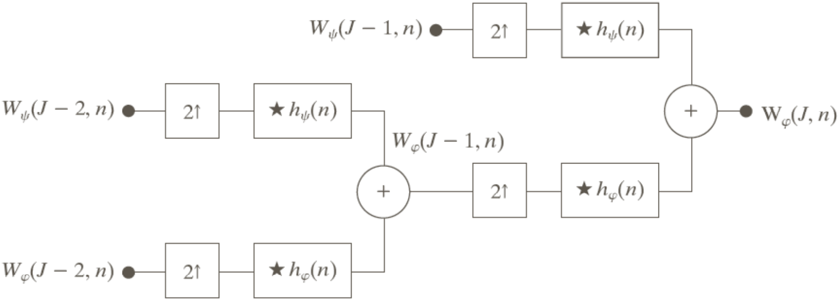


Figure: HAAR-synthesis (2 levels)

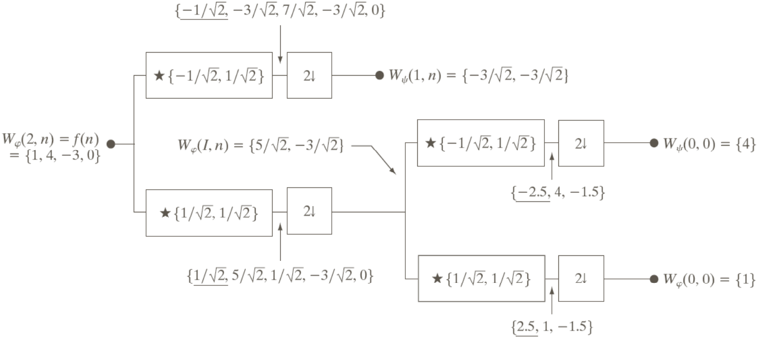


Figure: HAAR analysis (2 levels) – example

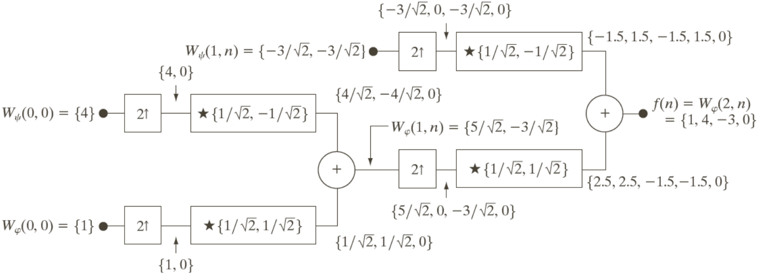


Figure: HAAR synthesis (2 levels) – example

Reminder (1)

- The 1D HAAR functions are

$$\text{scaling function} \qquad \phi(t) = \mathbf{1}_{[0,1)}(t)$$
$$\text{wavelet function} \qquad \psi(t) = \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t)$$

- The other functions are derived by using dilation and translation w.r.t. the dyadic intervals $I_{j,k}$ $(j, k \in \mathbb{Z})$:

$$\begin{aligned} \phi_{j,k}(t) &= 2^{j/2}\, \mathbf{1}_{I_{j,k}}(t) &&= 2^{j/2}\, \phi(2^j\, t - k) \\ \psi_{j,k}(t) &= 2^{j/2}\, \left(\mathbf{1}_{I_{j+1,2k}}(t) - \mathbf{1}_{I_{j+1,2k+1}}(t)\right) &&= 2^{j/2}\, \psi(2^j\, t - k) \end{aligned}$$

Reminder (2)

- Using $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$) one has

$$\begin{bmatrix} \phi \\ \psi \end{bmatrix} = H \cdot \begin{bmatrix} \phi_{1,0} \\ \phi_{1,1} \end{bmatrix} \quad , \quad \begin{bmatrix} \phi_{1,0} \\ \phi_{1,1} \end{bmatrix} = H \cdot \begin{bmatrix} \phi \\ \psi \end{bmatrix}$$

- and thus for all $j, k \in \mathbb{Z}$

$$\begin{bmatrix} \phi_{j,k} \\ \psi_{j,k} \end{bmatrix} = H \cdot \begin{bmatrix} \phi_{j+1,2k} \\ \phi_{j+1,2k+1} \end{bmatrix} \quad , \quad \begin{bmatrix} \phi_{j+1,2k} \\ \phi_{j+1,2k+1} \end{bmatrix} = H \cdot \begin{bmatrix} \phi_{j,k} \\ \psi_{j,k} \end{bmatrix}$$

2D HAAR functions (1)

- The 2D HAAR functions are the four functions

$$\begin{aligned} \phi(x,y) &= \phi(x)\cdot\phi(y) \\ \psi^H(x,y) &= \psi(x)\cdot\phi(y) \\ \psi^V(x,y) &= \phi(x)\cdot\psi(y) \\ \psi^D(x,y) &= \psi(x)\cdot\psi(y) \end{aligned}$$

  - ϕ is the *2D* HAAR *scaling function*
  - the ψ^H, ψ^V, ψ^D are the *2D* HAAR *wavelet functions*
- Suggestively: "*H*" stands for *horizontal*, ' '*V*" for *vertical*, and "*D*" für *diagonal*, corresponding to the directions in which these functions register changes

2D HAAR functions (2)

- Obviously

$$\begin{bmatrix} \phi \\ \psi^H \\ \psi^V \\ \psi^D \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{1,0,0} \\ \phi_{1,1,0} \\ \phi_{1,0,1} \\ \phi_{1,1,1} \end{bmatrix}$$

$$= (H \otimes H) \begin{bmatrix} \phi_{1,0,0} \\ \phi_{1,1,0} \\ \phi_{1,0,1} \\ \phi_{1,1,1} \end{bmatrix}$$

- $H \otimes H$ is again an orthogonal matrix

2D HAAR functions (3)

- For any a, b, c, d one has

$$(H \otimes H)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix} \quad \Longleftrightarrow \quad H \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot H = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

2D HAAR functions (4)

- By dilation and translation one generates the 2D HAAR functions for $j, k, \ell \in \mathbb{Z}$:

$$\begin{aligned}
\phi_{j,k,\ell}(x,y) &= \phi_{j,k}(x) \cdot \phi_{j,\ell}(y) = 2^j \phi(2^j x - k, 2^j y - \ell) \\
\psi^H_{j,k,\ell}(x,y) &= \psi_{j,k}(x) \cdot \phi_{j,\ell}(y) = 2^j \psi^H(2^j x - k, 2^j y - \ell) \\
\psi^V_{j,k,\ell}(x,y) &= \phi_{j,k}(x) \cdot \psi_{j,\ell}(y) = 2^j \psi^V(2^j x - k, 2^j y - \ell) \\
\psi^D_{j,k,\ell}(x,y) &= \psi_{j,k}(x) \cdot \psi_{j,\ell}(y) = 2^j \psi^D(2^j x - k, 2^j y - \ell)
\end{aligned}$$

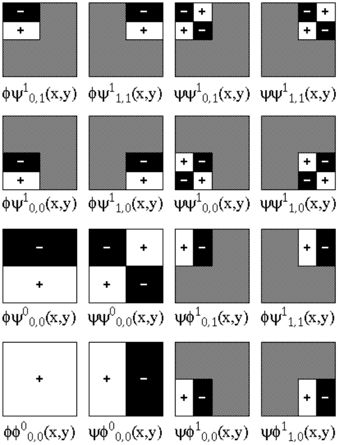
2D HAAR functions (5)

- 2D scaling equations and wavelet equations written in matrix form:

$$\begin{bmatrix} \phi_{j,k,\ell} & \psi^H_{j,k,\ell} \\ \psi^V_{j,k,\ell} & \psi^D_{j,k,\ell} \end{bmatrix} = H \cdot \begin{bmatrix} \phi_{j+1,2k,2\ell} & \phi_{j+1,2k+1,2\ell} \\ \phi_{j+1,2k,2\ell+1} & \phi_{j+1,2k+1,2\ell+1} \end{bmatrix} \cdot H$$

- Equivalently

$$\begin{bmatrix} \phi_{j,k,\ell} \\ \psi^H_{j,k,\ell} \\ \psi^V_{j,k,\ell} \\ \psi^D_{j,k,\ell} \end{bmatrix} = (H \otimes H) \begin{bmatrix} \phi_{j+1,2k,2\ell} \\ \phi_{j+1,2k+1,2\ell} \\ \phi_{j+1,2k,2\ell+1} \\ \phi_{j+1,2k+1,2\ell+1} \end{bmatrix}$$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ψϕ¹1,0(x,y)

Vector spaces (1)

- The vector spaces relevant for 2D wavelet analysis and synthesis are:

$$\begin{aligned}\mathcal{V}_j &= \overline{span}\left\{\phi_{j,k,\ell}\right\}\\ \mathcal{W}_j^H &= \overline{span}\left\{\psi^H_{j,k,\ell}\right\}\\ \mathcal{W}_j^V &= \overline{span}\left\{\psi^V_{j,k,\ell}\right\}\\ \mathcal{W}_j^D &= \overline{span}\left\{\psi^D_{j,k,\ell}\right\}\end{aligned}$$

- For $\mathcal{L}^2(\mathbb{R}^2)$ HAAR wavelets take all indices $j, k, \ell \in \mathbb{Z}$
- For $\mathcal{L}^2([0,1]^2)$ HAAR wavelets take indices $j \geq 0, 0 \leq k, \ell < 2^j$
- Spaces $\mathcal{V}_j$ are the *approximation spaces*,
  spaces $\mathcal{W}_j^H, \mathcal{W}_j^V, \mathcal{W}_j^D$ are the *detail* spaces or *wavelet spaces*

Vector spaces (2)

- The results about complete bases for the $\mathcal{L}^2$ spaces carry over to the 2D situation without problems. Similarly one has the corresponding identities for the wavelet coefficients
- For any $j \in \mathbb{Z}$ one has

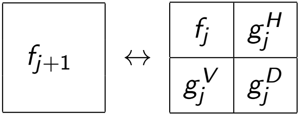
$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j^H \oplus \mathcal{W}_j^V \oplus \mathcal{W}_j^D$$

- which says: any function $f \in \mathcal{V}_{j+1}$ has a unique orthogonal decomposition

$$f_{j+1} = f_j + g_j^H + g_j^V + g_j^D \quad \text{with } f_j \in \mathcal{V}_j,\ g_j^x \in \mathcal{W}_j^x \quad (x \in \{H, D, V\})$$

Vector spaces (3)

- For HAAR wavelet analysis in $[0,1]^2$ one ranges the coefficients of these functions w.r.t. the bases in the respective subspaces with side length 2^{j+1}:



- One phase of HAAR *wavelet analysis* consists in computing the data on the right from the data on the left
- One level of HAAR-*wavelet synthesis* consists in computing the data on the left from the data on the right

Analysis

- A (discrete) *image* is a $(2m \times 2n)$ matrix $\mathbf{a}_{2m,2n}$ (of gray values, say)
- One phase of wavelet analysis replaces this image by four $(m \times n)$ images $\mathbf{a}'_{m,n}$, $\mathbf{d}^H_{m,n}$, $\mathbf{d}^V_{m,n}$, $\mathbf{d}^D_{m,n}$ following the scheme

$$\boxed{\mathbf{a}_{2m,2n}} \rightarrowtail \begin{array}{|c|c|}\hline \mathbf{a}'_{m,n} & \mathbf{d}^H_{m,n} \\ \hline \mathbf{d}^V_{m,n} & \mathbf{d}^D_{m,n} \\ \hline \end{array}$$

- Again: **a** stands for "approximation" and **d** stands for "detail".
- for level-k Haar analysis it is required that the side lengths are multiples if 2^k

Transformation matrices

- The transformation can be conveniently described using the matrices used in the 1D case. Let

$$A_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & \\ 1 & & & \\ & 1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 1 \end{bmatrix} \quad D_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ & & \ddots & \\ & & & 1 \\ & & & -1 \end{bmatrix}.$$

These are matrices of format $(2n \times n)$

Analysis as a matrix operation (1)

- Then

$$\boxed{\mathbf{a}_{2m,2n}} \rightarrowtail \begin{array}{|c|c|}\hline \mathbf{a}'_{m,n} & \mathbf{d}^H_{m,n} \\ \hline \mathbf{d}^V_{m,n} & \mathbf{d}^D_{m,n} \\ \hline \end{array} = \begin{bmatrix} A^\dagger_m \\ D^\dagger_m \end{bmatrix} \cdot \mathbf{a}_{2m,2n} \cdot \begin{bmatrix} A_n & D_n \end{bmatrix}$$

- Written in full detail:

$$\boxed{\begin{aligned} \mathbf{a}'_{m,n} &= A^\dagger_m \cdot \mathbf{a}_{2m,2n} \cdot A_n \\ \mathbf{d}^H_{m,n} &= A^\dagger_m \cdot \mathbf{a}_{2m,2n} \cdot D_n \\ \mathbf{d}^V_{m,n} &= D^\dagger_m \cdot \mathbf{a}_{2m,2n} \cdot A_n \\ \mathbf{d}^D_{m,n} &= D^\dagger_m \cdot \mathbf{a}_{2m,2n} \cdot D_n \end{aligned}}$$

Analysis as a matrix operation (2)

- One can and one should read this as follows:

  The 2D Haar transform executed on an image $\mathbf{a}_{2m,2n}$ consists in

  - first executing the 1D Haar transform on the rows of $\mathbf{a}_{2m,2n}$ (in parallel), which gives

$$\widetilde{\mathbf{a}}_{2m,2n} = \mathbf{a}_{2m,2n} \cdot \begin{bmatrix} A_n & D_n \end{bmatrix};$$

  - then executing the 1D Haar transform on the columns of $\widetilde{\mathbf{a}}_{2m,2n}$ (in parallel), which gives

$$\begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix} \cdot \widetilde{\mathbf{a}}_{2m,2n} = \begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix} \cdot \mathbf{a}_{2m,2n} \cdot \begin{bmatrix} A_n & D_n \end{bmatrix}.$$

- One can do it also the other way round: first acting on the columns and then on the rows. The result is the same

Synthesis as a matrix operation

- For synthesis the above relation has to be inverted, which is no problem at all because of the orthogonality of the matrices $\begin{bmatrix} A_n & D_n \end{bmatrix}$:

$$\mathbf{a}_{2m,2n} = \begin{bmatrix} A_m & D_m \end{bmatrix} \cdot \begin{array}{|c|c|} \hline \mathbf{a}'_{m,n} & \mathbf{d}^H_{m,n} \\ \hline \mathbf{d}^V_{m,n} & \mathbf{d}^D_{m,n} \\ \hline \end{array} \cdot \begin{bmatrix} A_n^\dagger \\ D_n^\dagger \end{bmatrix}$$

- Written explicitly:

$$\boxed{\mathbf{a}_{2m,2n} = A_m \cdot \mathbf{a}'_{m,n} \cdot A_n^\dagger + D_m \cdot \mathbf{d}^V_{m,n} \cdot A_n^\dagger + A_m \cdot \mathbf{d}^H_{m,n} \cdot D_n^\dagger + D_m \cdot \mathbf{d}^D_{m,n} \cdot D_n^\dagger}$$

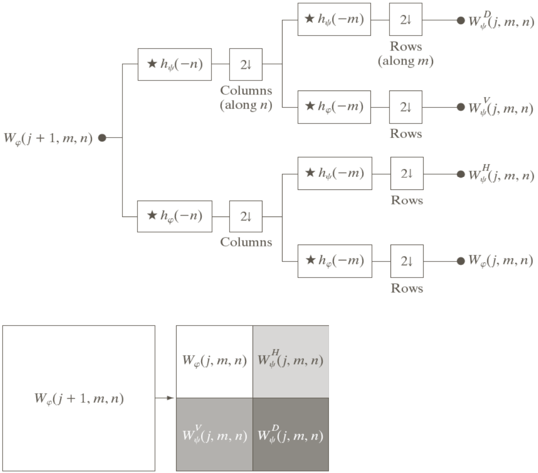


Figure: One-level 2D HAAR WT as a filter bank (analysis)

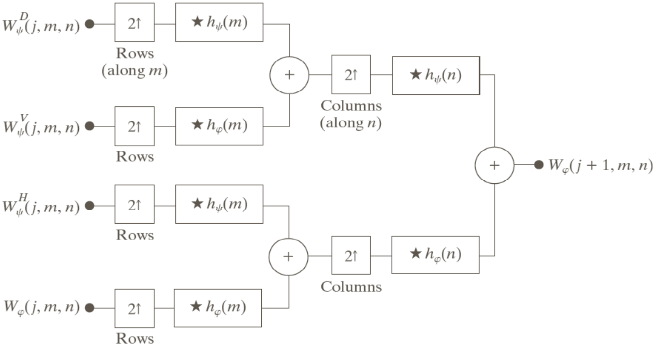


Figure: One level 2D Haar WT as a filter bank (synthesis)

2D multilevel Haar transform

- The 2D Haar transform can be extended to a transformation running over several levels by iteratively applying the very same procedure to the arrays of approximation coefficients generated. This scheme applies to other wavelet transforms as well

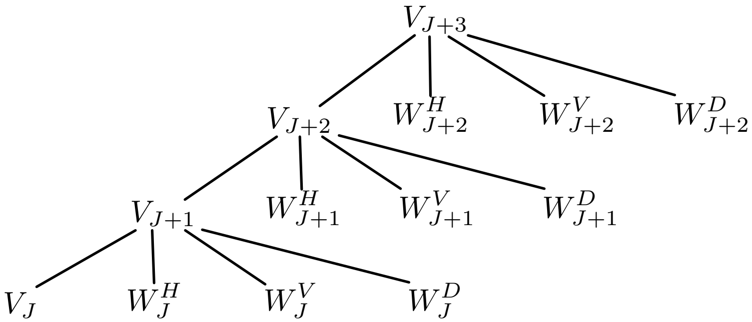


Figure: Decomposition scheme for a 2D-3-level-WT

2D multilevel Haar transform

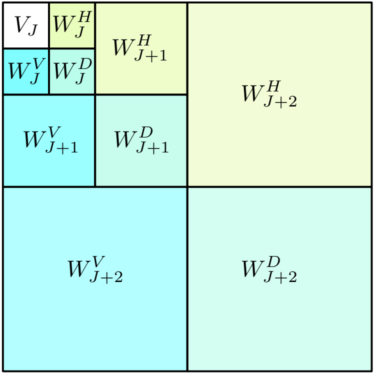


Figure: Coefficient scheme for a 2D-3-level-WT