Estimating the size of Haar wavelet coefficients

The behavior of the size of wavelet coefficents over several levels of resolution indicates the presence or absence of jump discontinuities

November 10, 2015

- Recall dyadic intervals

$$I_{j,k} = [\ 2^{-j}k,\ 2^{-j}(k+1)\ ) \qquad (j, k \in \mathbb{Z})$$

  with midpoints $t_{j,k} = 2^{-j}(k + 1/2)$

- Theorem:
  1. If a function f is twice differentiable on $I_{j,k}$, then the Haar wavelet coefficients satisfy

  $$|\langle\, f \mid \psi_{j,k} \,\rangle| \approx \frac{1}{4}\,|\, f'(t_{j,k})\,|\, 2^{-3j/2}$$

  2. If a function f is twice differentiable on $I_{j,k}$, except for a jump discontinuity at $t_0 \in I_{j,k}$, then the Haar wavelet coefficients satisfy

  $$|\langle\, f \mid \psi_{j,k} \,\rangle| \approx \frac{1}{4}\,|\, f(t_0^-) - f(t_0^+)\,|\, 2^{-j/2}$$

- Consequence: Jump discontinuities of an otherwise differentiable function can be recognized from the size of the wavelet coefficients over serveral levels of resolution!

- Proof of 1.
  - Use Taylor's expansion around $t_{j,k}$ for $t \in I_{j,k}$:

$$f(t) = f(t_{j,k}) + f'(t_{j,k})(t - t_{j,k}) + \frac{1}{2} f''(\xi_{j,k})(t - t_{j,k})^2$$

    for some $\xi_{j,k} = \xi_{j,k}(t) \in I_{j,k}$
  - Now compute the wavelet coefficients

$$\begin{aligned} \langle f \mid \psi_{j,k} \rangle &= \int_{I_{j,k}} f(t)\, \psi_{j,k}(t)\, dt \\ &= f(t_{j,k}) \int_{I_{j,k}} \psi_{j,k}(t)\, dt \\ &\quad + f'(t_{j,k}) \int_{I_{j,k}} (t - t_{j,k})\, \psi_{j,k}(t)\, dt \\ &\quad + \frac{1}{2} \int_{I_{j,k}} f''(\xi_{j,k})(t - t_{j,k})^2\, \psi_{j,k}(t)\, dt \end{aligned}$$

- and use the following facts
  - $\int_{I_{j,k}} \psi_{j,k}(t)\, dt = 0$
  - $\int_{I_{j,k}} t\, \psi_{j,k}(t)\, dt = -\frac{1}{4}\, 2^{-3j/2}$
  - $\int_{I_{j,k}} (t - t_{j,k})^2\, dt = \frac{1}{12}\, 2^{-3j}$
  - and hence

$$\left| \int_{I_{j,k}} f''(\xi_{j,k})\,(t - t_{j,k})^2\, \psi_{j,k}(t)\, dt \right|$$

$$\leq \max_{t \in I_{j,k}} |f''(t)| \int_{I_{j,k}} \left| (t - t_{j,k})^2\, \psi_{j,k}(t) \right|\, dt$$

$$\leq \max_{t \in I_{j,k}} |f''(t)| \cdot 2^{j/2} \int_{I_{j,k}} (t - t_{j,k})^2\, dt$$

$$\leq \max_{t \in I_{j,k}} |f''(t)| \cdot \frac{1}{12} \cdot 2^{-5j/2}$$

  - which proves 1.

- Proof of 2.
  - Assume that $t_0 \in I^{\ell}_{j,k}$
  - Expand to the left and right of t_0

$$
\begin{aligned}
f(t) &= f(t_0^-) + f'(\xi_-)(t - t_0) \quad t \in [0, t_0),\ \xi_- \in (t, t_0) \\
f(t) &= f(t_0^+) + f'(\xi_+)(t - t_0) \quad t \in (t_0, 1],\ \xi_+ \in (t_0, t)
\end{aligned}
$$

  - For the wavelet coefficients obtain

$$
\langle\, f \,|\, \psi_{j,k} \,\rangle = \int_{I_{j,k}} f(t)\, \psi_{j,k}(t)\, dt =
$$

$$
2^{j/2} \underbrace{\left(\int_{2^{-j}k}^{t_0} f(t_{0^-}) + \int_{t_0}^{t_{j,k}} f(t_{0^+}) - \int_{t_{j,k}}^{2^{-j}(k+1)} f(t_{0^+}) \right)}_{(t_0 - 2^{-j}k) \cdot (f(t_0^-) - f(t_0^+))} + \varepsilon_{j,k}
$$

- where

$$|\varepsilon_{j,k}| \leq \max_{\substack{t\in I_{j,k}\\ t\neq t_0}} |f'(t)| \int_{I_{j,k}} |t - t_0|\, |\psi_{j,k}(t)|\, dt \leq \max(\ldots)\frac{1}{4}2^{-3j/2}$$

- Observing that $|t_0 - 2^{-j}k| \approx \frac{1}{4}\, 2^{-j}$ gives

$$\left| 2^{j/2} \left(\int_{2^{-j}k}^{t_0} f(t_{0-}) + \int_{t_0}^{t_{j,k}} f(t_{0^+}) - \int_{t_{j,k}}^{2^{-j}(k+1)} f(t_{0^+}) \right) \right|$$
$$= 2^{j/2} \left| t_0 - 2^{-j}k \right| \cdot \left| f(t_0^-) - f(t_0^+) \right| \approx \frac{1}{4} 2^{-j/2} \left| (f(t_0^-) - f(t_0^+)) \right|$$

- This proves 2.

- Case 1 (smooth): the $\langle\, f \mid \psi_{j,k} \,\rangle$ behave $\propto 2^{-3j/2}$
- Case 2 (jump): the $\langle\, f \mid \psi_{j,k} \,\rangle$ behave $\propto 2^{-j/2}$