Restoration of Missing Data in Limited Angle Tomography Based on Helgason-Ludwig Consistency Conditions

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Outline

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- Regression in Sinogram Domain (Chebyshev Fourier transform)
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III. Results and Discussion

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- Clinical data

IV. Conclusion

I. Motivation

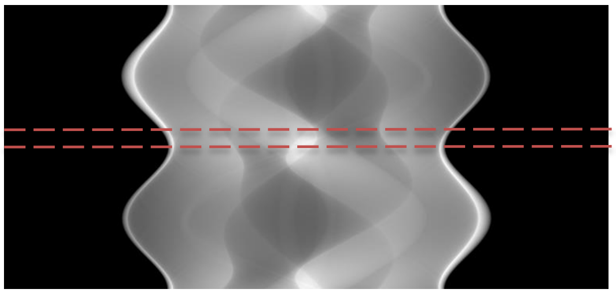
1. Limited angle tomography

- Classical incomplete data reconstruction problem
- Gantry rotation is restricted by other system parts or external objects
- Parallel-beam: less than 180º;

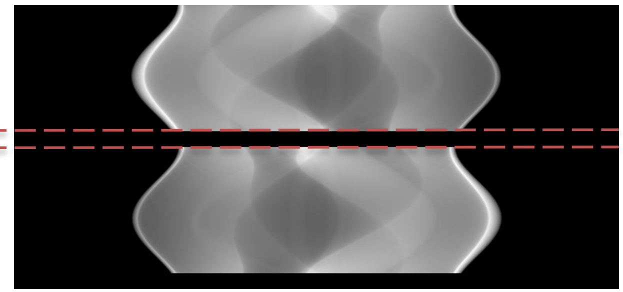
  Fan-beam and cone-beam: less than a short scan

Siemens Artis zee multi-purpose system

2. Parallel-beam limited angle tomography



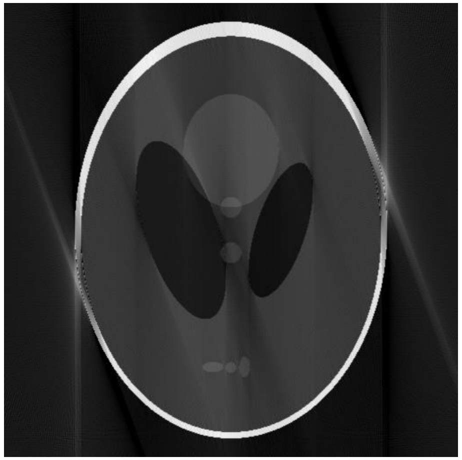
Complete sinogram



measured

Limited angle sinogram, 160º

Shepp-Logan phantom



Limited angle reconstruction, FBP

Method: Regression in Sinogram Domain

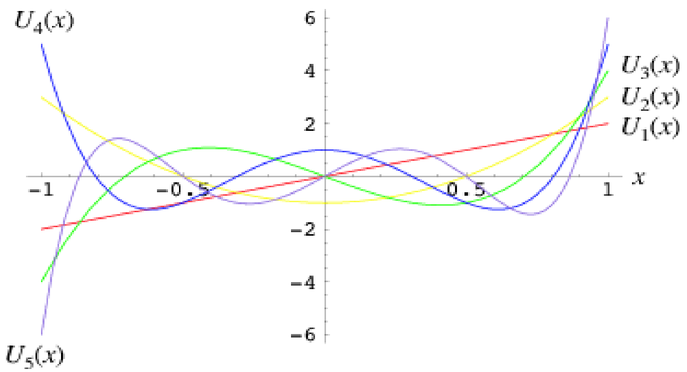
1. Chebyshev polynomials of the second kind

- Definition:

$$U_n(s) = \frac{\sin((n+1)\arccos(s))}{\sqrt{1-s^2}} \quad \left(= \frac{\sin((n+1)t)}{\sin(t)}, \text{if } s = \cos(t) \right)$$

- $U_n(s)$ is a polynomial of degree n with only even/odd monomials

$$U_0(s) = 1;$$
$$U_1(s) = 2s;$$
$$U_2(s) = 4s^2 - 1;$$
$$U_3(s) = 8s^3 - 4s;$$
...

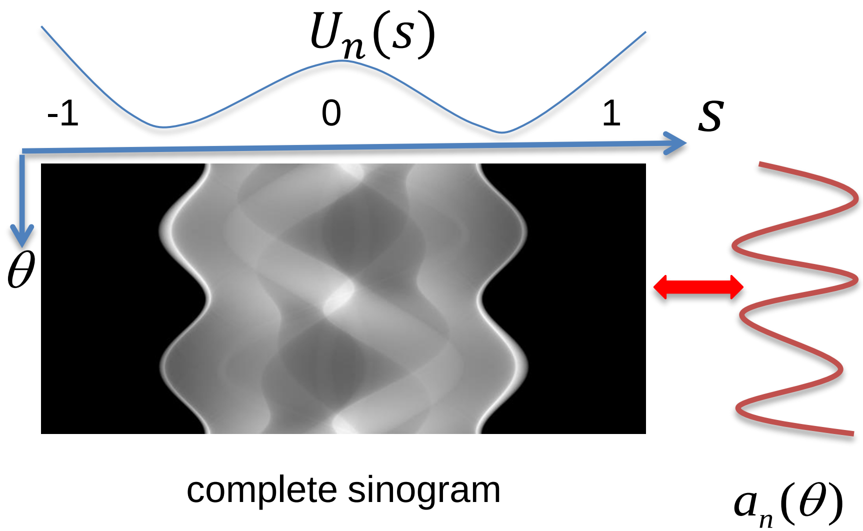


$U_n(s)$ (Wolfram MathWorld)

2. Moment curves

- n^{th} order moment curve of the parallel-beam sinogram $p(s,\theta)$

$$a_n(\theta) = \int_{-1}^{1} p(s,\theta) U_n(s) ds$$

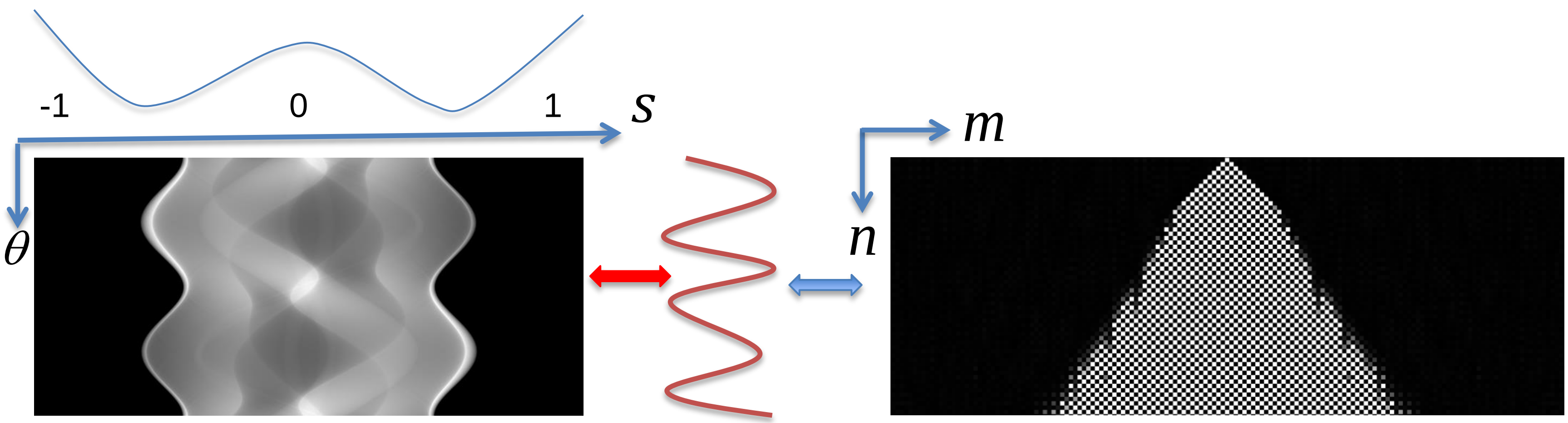


3. Fourier transform of the moment curve

- Fourier transform:

$$a_{n,m} = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} a_n(\theta) d\theta$$

- **HLCC:** $a_{n,m} = 0 \quad |m| > n \text{ or } n + |m| \text{ is odd}$



Complete sinogram

$a_n(\theta)$

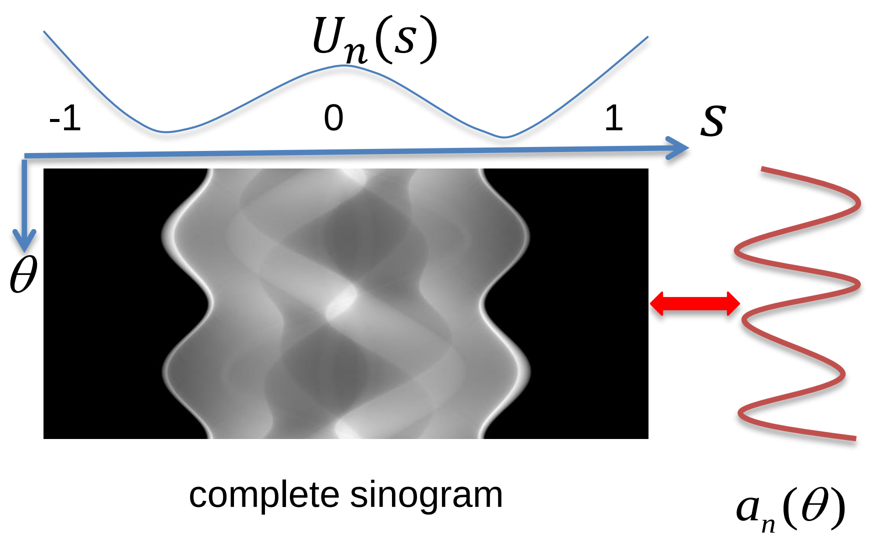
Checkerboard pattern, $a_{n,m}$

4. Invertible transform

- Orthogonal set with weight $W(s) = (1 - s^2)^{1/2}$:

$$\int_{-1}^{1} W(s) U_n(s) U_m(s) ds = \begin{cases} 0, & n \neq m \\ \pi/2, & n = m \end{cases}$$

- Sinogram restoration: $p(s,\theta) = \frac{2}{\pi} \sum_{0}^{\infty} a_n(\theta)(W(s)U_n(s))$



5. Sinogram restoration, complete data

- Restoration of sinogram using $0 - n_r$ orders

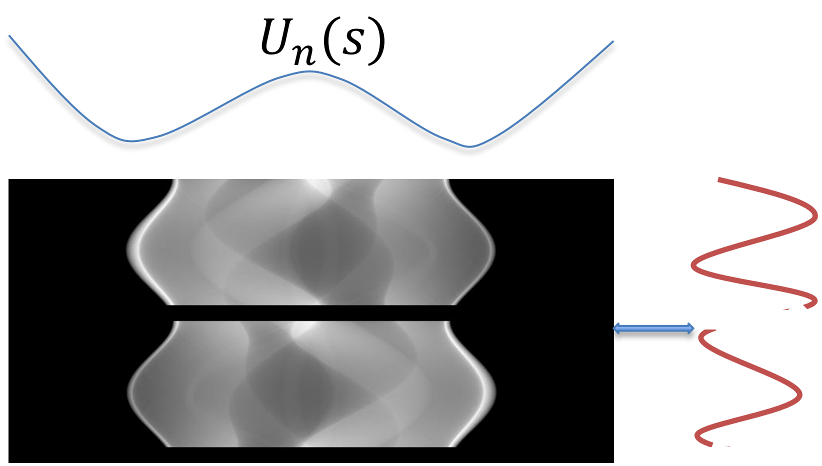
$$p_{n_r}(s,\theta) = \frac{2}{\pi}\sum_{0}^{n_r} a_n(\theta)(W(s)U_n(s))$$

Restored sinograms when the number of orders n_r increases

6. nth moment curve, limited angle data

- Chebyshev transform, n^{th} moment curve:

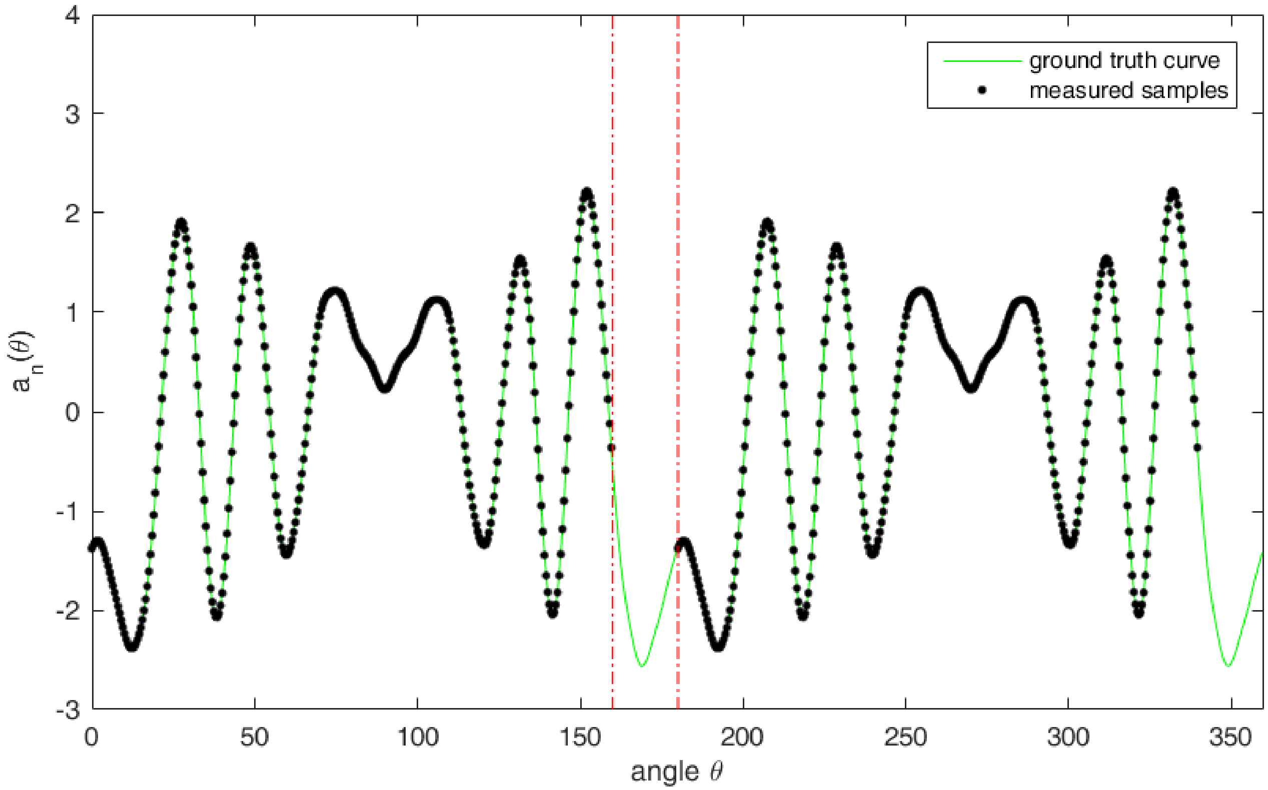
$$a_n(\theta) = \int_{-1}^{1} p(s,\theta) U_n(s) ds$$



Limited angle parallel-beam sinogram,
160º measured

$a_n(\theta)$

7. Regression problem



Curve fitting problem, n = 100

8. Analytical Form of $a_n(\theta)$

- Based on HLCC, the analytical form of $a_n(\theta)$ is known

- If n is even:

$$a_n(\theta) = b_0 + b_2\cos(2\theta) + c_2\sin(2\theta) + b_4\cos(4\theta) + c_4\sin(4\theta) + \ldots + b_n\cos(n\theta) + c_n\sin(n\theta)$$

- If n is odd:

$$a_n(\theta) = b_1\cos(\theta) + c_1\sin(\theta) + b_3\cos(3\theta) + c_3\sin(3\theta) + \ldots + b_n\cos(n\theta) + c_n\sin(n\theta)$$

- In both cases, $n + 1$ unknown coefficients

9. Limit angle regression problem

- When n is even:

$$\begin{bmatrix} 1 & \cos(2\theta_0) & \sin(2\theta_0) & \cos(4\theta_0) & \sin(4\theta_0) & \dots & \cos(n\theta_0) & \sin(n\theta_0) \\ 1 & \cos(2\theta_1) & \sin(2\theta_1) & \cos(4\theta_1) & \sin(4\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ 1 & \cos(2\theta_2) & \sin(2\theta_2) & \cos(4\theta_2) & \sin(4\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \cdots \\ 1 & \cos(2\theta_{N-1}) & \sin(2\theta_{N-1}) & \cos(4\theta_{N-1}) & \sin(4\theta_{N-1}) & \dots & \cos(n\theta_{N-1}) & \sin(n\theta_{N-1}) \end{bmatrix} \begin{bmatrix} b_0 \\ b_2 \\ c_2 \\ b_4 \\ c_4 \\ \cdots \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} a_n(\theta_0) \\ a_n(\theta_1) \\ a_n(\theta_2) \\ \vdots \\ a_n(\theta_{N-1}) \end{bmatrix}$$

where N is the number of projections

- When n is odd:

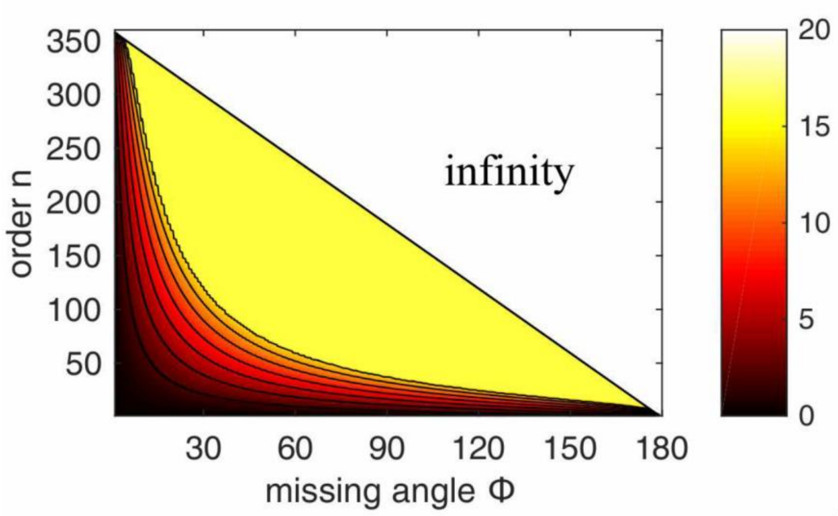
$$\begin{bmatrix} \cos(\theta_0) & \sin(\theta_0) & \cos(3\theta_0) & \sin(3\theta_0) & \dots & \cos(n\theta_0) & \sin(n\theta_0) \\ \cos(\theta_1) & \sin(\theta_1) & \cos(3\theta_1) & \sin(3\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ \cos(\theta_2) & \sin(\theta_2) & \cos(3\theta_2) & \sin(3\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \cdots \\ \cos(\theta_{N-1}) & \sin(\theta_{N-1}) & \cos(3\theta_{N-1}) & \sin(3\theta_{N-1}) & \dots & \cos(n\theta_{N-1}) & \sin(n\theta_{N-1}) \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ b_3 \\ c_3 \\ \cdots \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} a_n(\theta_0) \\ a_n(\theta_1) \\ a_n(\theta_2) \\ \vdots \\ a_n(\theta_{N-1}) \end{bmatrix}$$

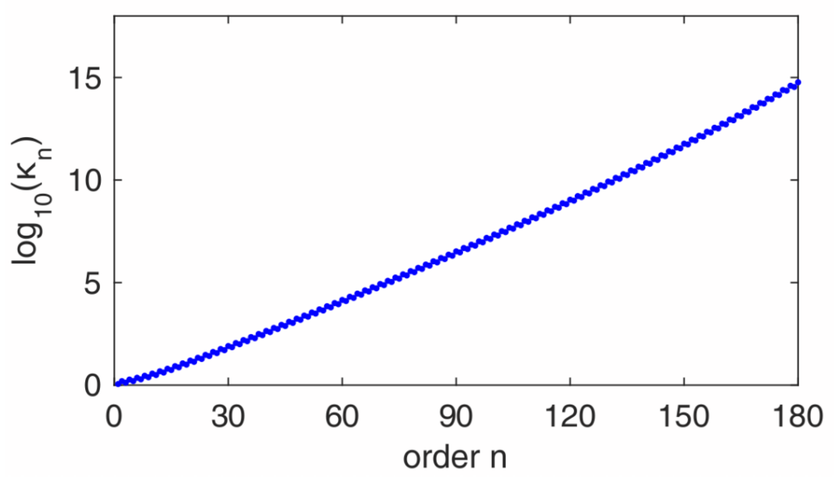
9. Ill-posedness of limited angle tomography

- Regression problem[1,2]:

$$X\beta = y$$

- Advantage: convenient to analyze its ill-posedness





[1] Louis A K & Törnig W 1980 Picture reconstruction from projections in restricted range. *Mathematical Methods in the Applied Sciences*
[2] Louis A K & Natterer F1983 Mathematical problems of computerized tomography. *Proceedings of the IEEE*

10. Ill-posed regression problem

- Various existing algorithms to solve ill-posed regression problems

- Lasso regression:

$$\boldsymbol{\beta} = \arg\min \tfrac{1}{2} \| \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{y} \|_2^2 + \lambda \| \boldsymbol{\beta} \|_1$$

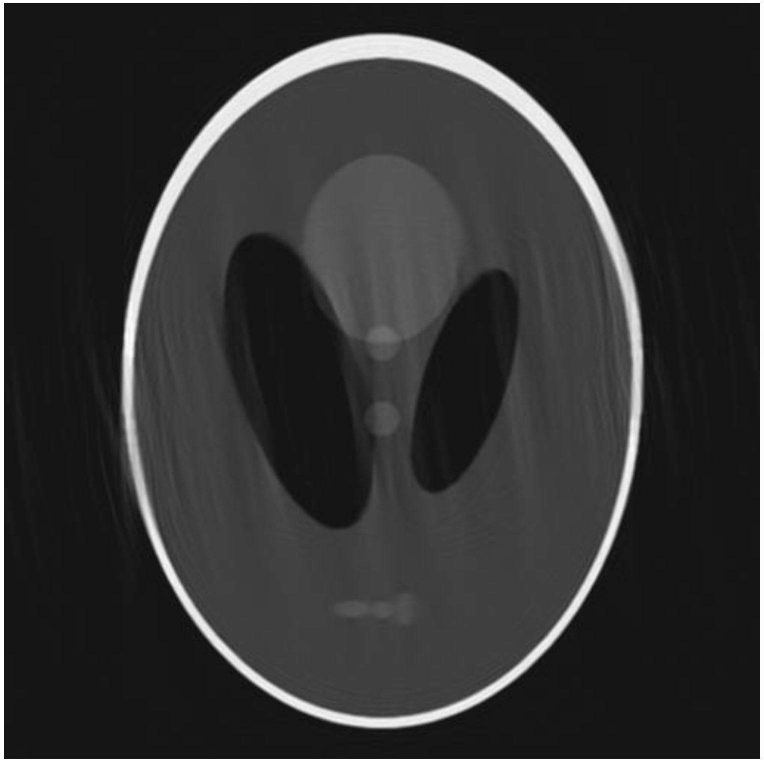
  solved by the iterative soft thresholding algorithm[3]

- Only low orders are estimated correctly, regression errors in higher orders cause artifacts

[3] Ingrid Daubechies, Michel Defrise, and Christine De Mol 2004 An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. Communications on pure and applied mathematics.

11. Artifacts caused by regression errors

- The regression artifacts appear as small streaks



Recondtrution of restored sinogram, f_{HLCC}

Method: Fusion in Frequency Domain

1. Fourier transform of $U_n(s)$

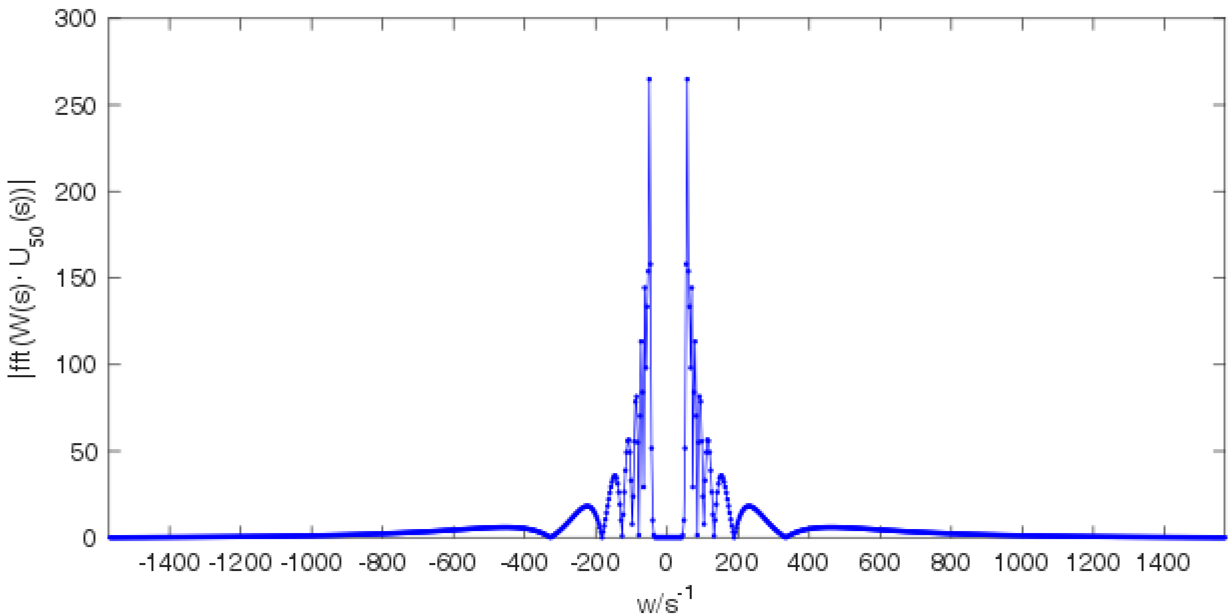
$$
\begin{aligned}
&\mathrm{F}(W(s)\cdot U_n(s))(w)\\
&=\int_{-1}^{1}\sqrt{1-s^2}\cdot\frac{\sin((n+1)\arccos(s))}{\sqrt{1-s^2}}\cdot e^{-iws}\mathrm{d}s \quad (s=\cos t)\\
&=-\int_{-\pi}^{\pi}\sin t\cdot\frac{\sin((n+1)t)}{\sin t}\cdot e^{-iw\cos t}\cdot\sin t\mathrm{d}t\\
&=-\int_{-\pi}^{\pi}\sin t\cdot\sin((n+1)t)\cdot e^{-iw\cos t}\mathrm{d}t\\
&=\int_{0}^{\pi}\cos((n+2)t)e^{-iw\cos t}\mathrm{d}t-\int_{0}^{\pi}\cos(nt)e^{-iw\cos t}\mathrm{d}t\\
&=\pi(J'_{n+2}(iw)-J'_{n}(iw))
\end{aligned}
$$

where $J'_n(z)$ is the modified Bessel function:

$$
J'_n(z)=\frac{1}{\pi}\int_{0}^{\pi}\cos(nt)e^{-z\cos t}\mathrm{d}t
$$

2. Fourier transform of $U_n(s)$ examples

- A Bessel function rapidly tends to zero when the argument becomes less than the order n



FFT of $U_{50}(s)$

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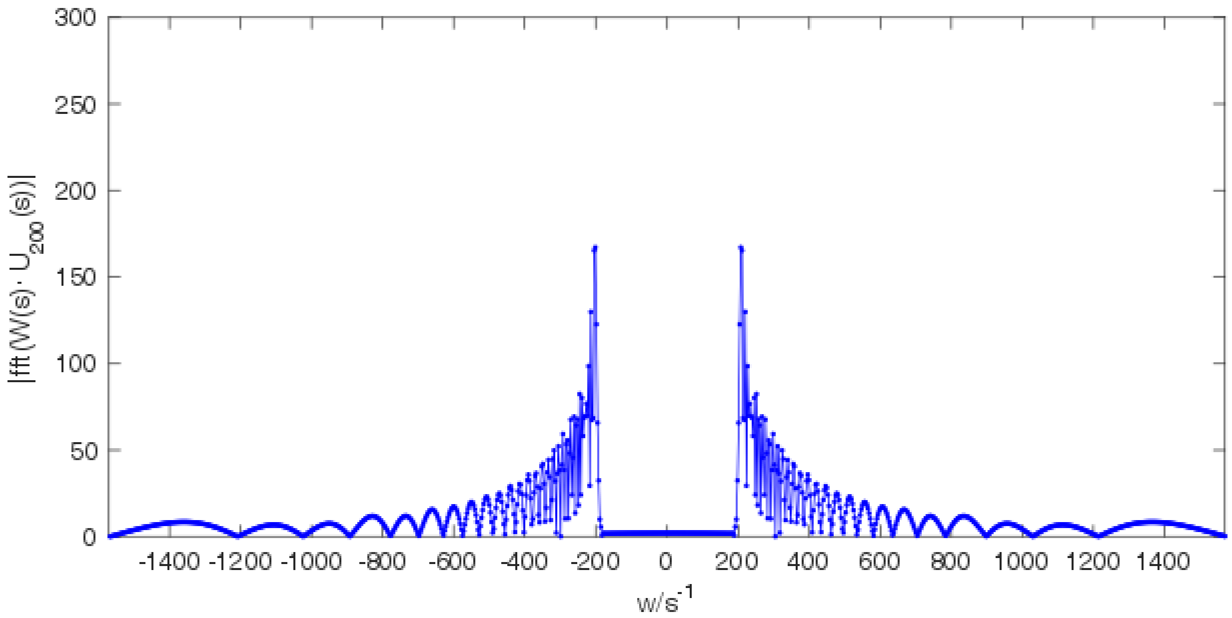
2. Fourier transform of $U_n(s)$ examples

- A Bessel function rapidly tends to zero when the argument becomes less than the order n



FFT of $U_{200}(s)$

3. High-pass filter

- $U_n(s) \cdot W(s)$ can be regarded as a high-pass filter

$$F(W(s)\cdot U_n(s))(w)\begin{cases}\approx 0, & 0 \le w \le w_{c,n}, \\ \ge 0, & w_{L,n} < w < w_{H,n}\end{cases}$$

where $w_{c,n} \approx n$ rad/s

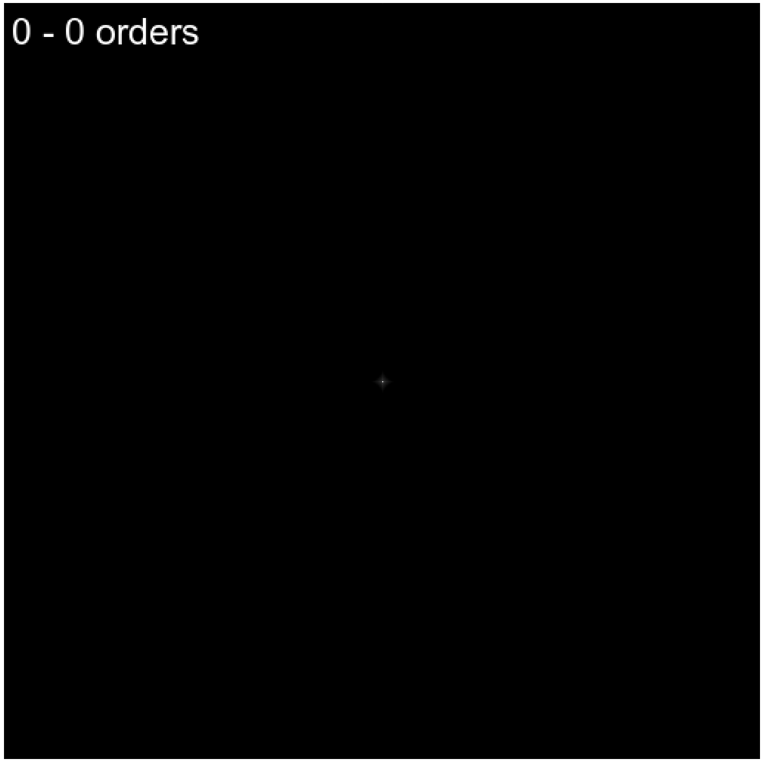
- Sinogram restoration:

$$p_{n_r}(s,\theta) = \frac{2}{\pi}\sum_{0}^{\infty} a_n(\theta)(W(s)U_n(s))$$

- the missing orders higher than n_r only contribute to the frequency range above w_{c,n_r}
- the frequency range [0, w_{c,n_r}] should be complete
- a circular area with radius w_{c,n_r} is restored

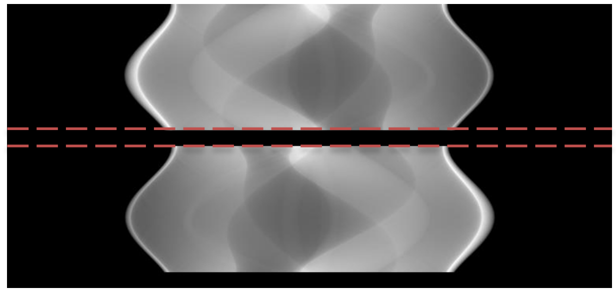
4. Frequency domain representation

- More orders, more high frequency components, less ringing artifacts



Fourier representation and reconstructed images when the number of orders n_r increases

5. Missing double wedge region

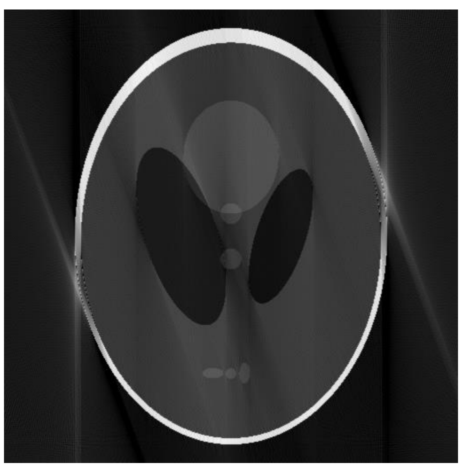


measured

Limited angle sinogram, 160º

FBP

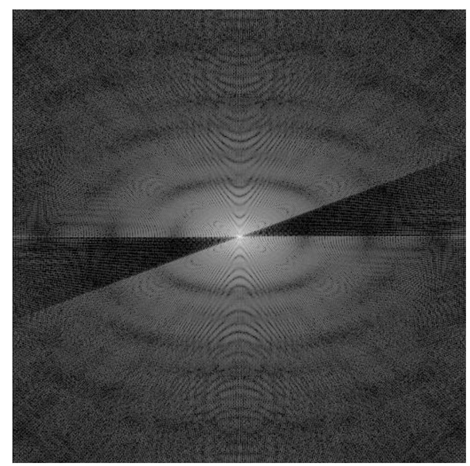
central slice theorem



Limited angle reconstruction

2D FFT

2D IFFT

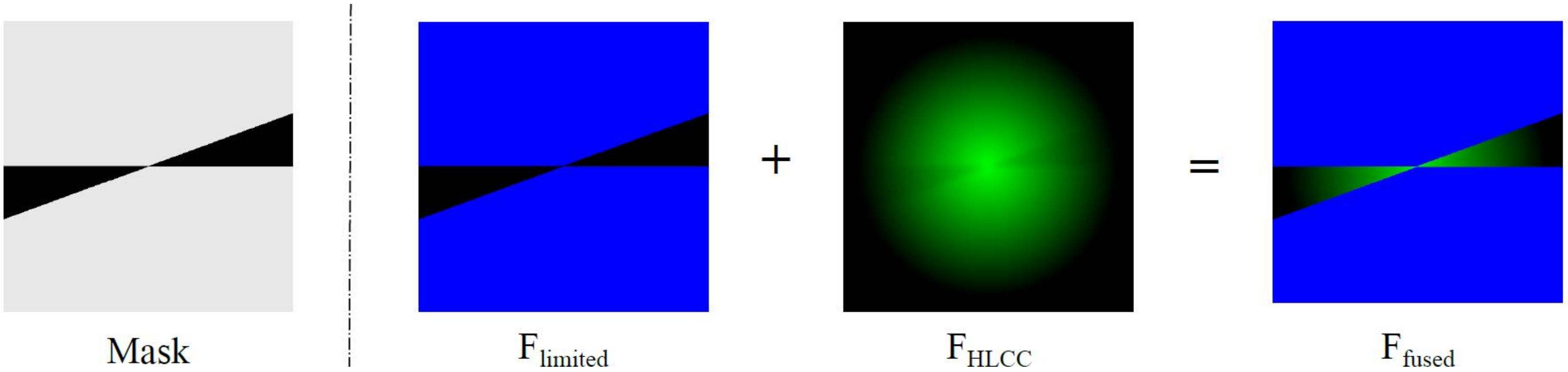


Frequency components of the reconstruction

6. Fusion in frequency domain

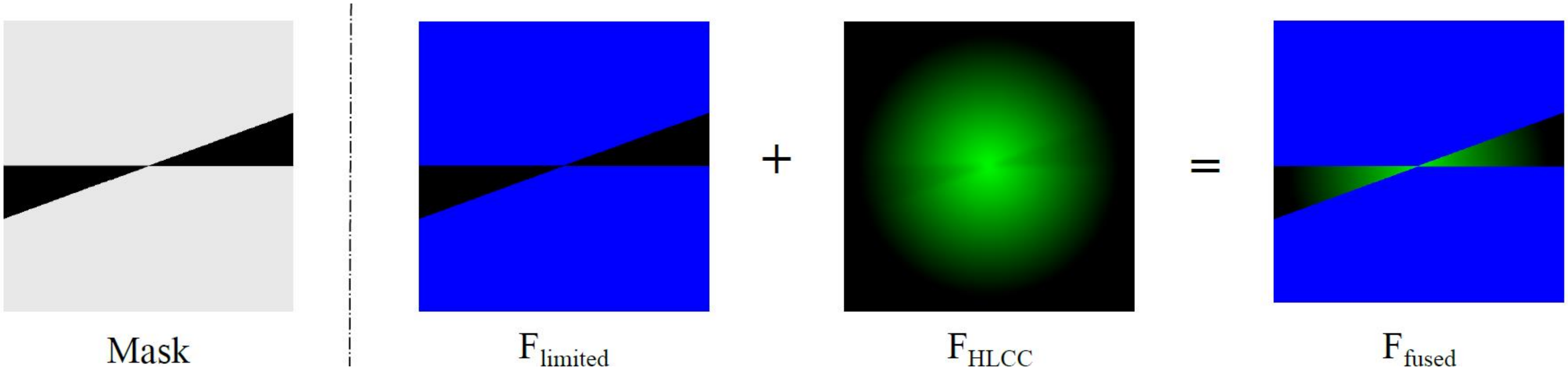
$$F_{\text{fused}}(w,\theta) = F_{\text{limited}}(w,\theta) \cdot M(w,\theta) + F_{\text{HLCC}}(w,\theta) \cdot (1 - M(w,\theta))$$

- Use the restored frequency components to fill in the missing double wedge region only



The black, blue, and green areas are the missing, measured, and HLCC estimated frequency components, respectively, where the faded green area might be not correctly estimated.

7. Bilateral filtering in spatial domain before fusion



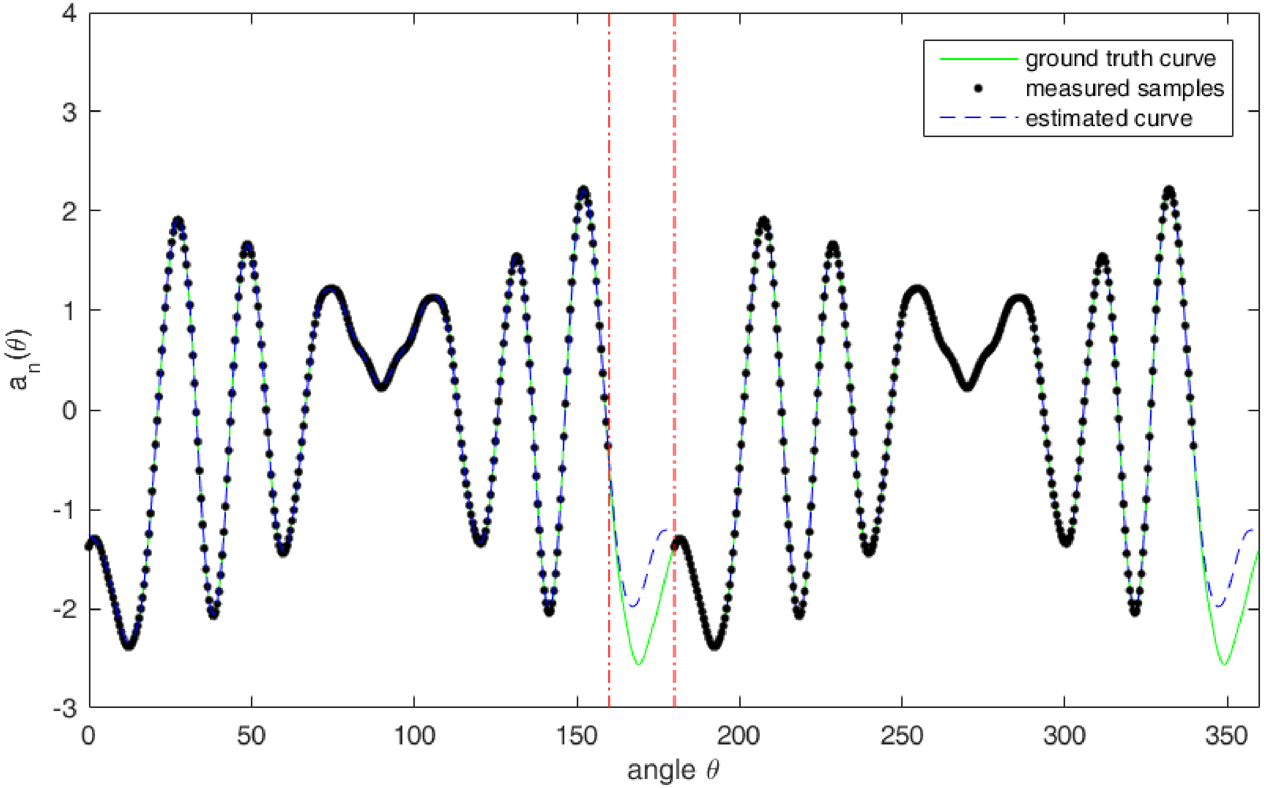
- High frequency components inside the double wedge region are still missing or erroneously estimated
- A strong bilateral filter can reduce regression artifacts and partially recover high frequency components associated with reliable high contrast edges

$$F_{\text{fused2}}(w,\theta) = F_{\text{limited}}(w,\theta) \cdot M(w,\theta) + F_{\text{HLCC,BF}}(w,\theta) \cdot (1 - M(w,\theta))$$

III. Results and Discussion

a. Shepp-Logan phantom, noise-free

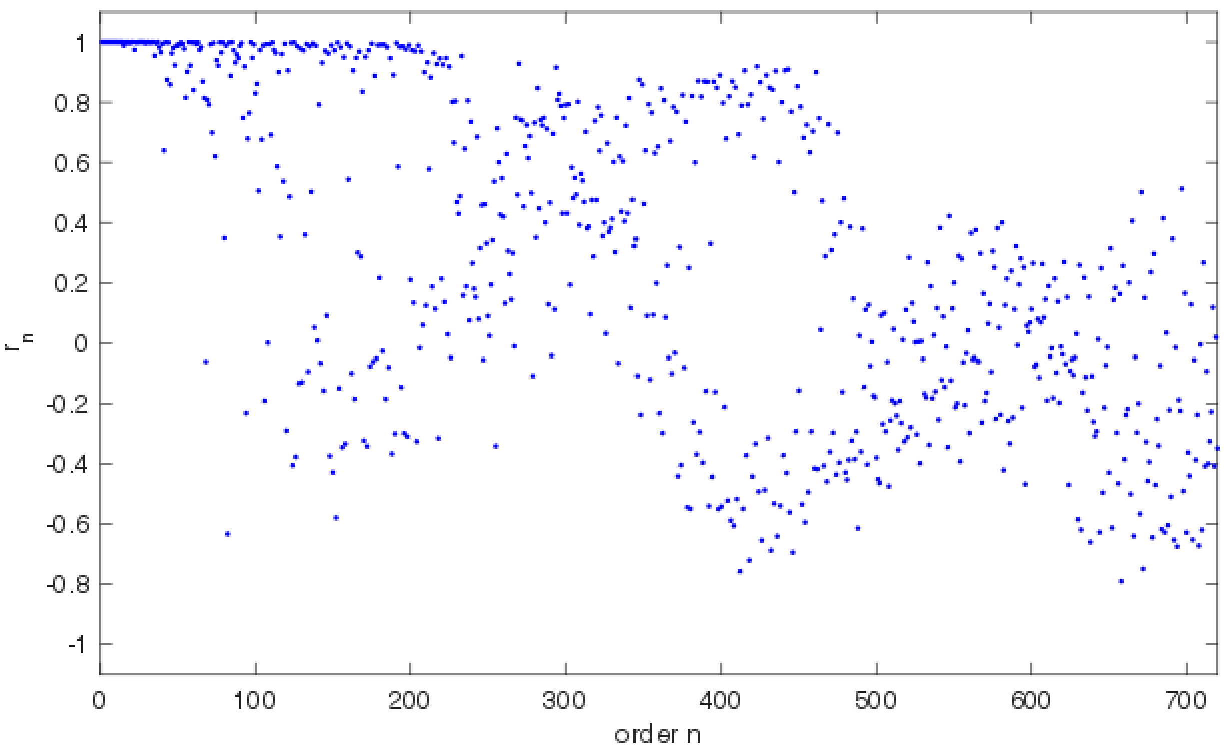
1. Estimation of moment curves



Estimated moment curve, e.g. n = 100, r = 0.86

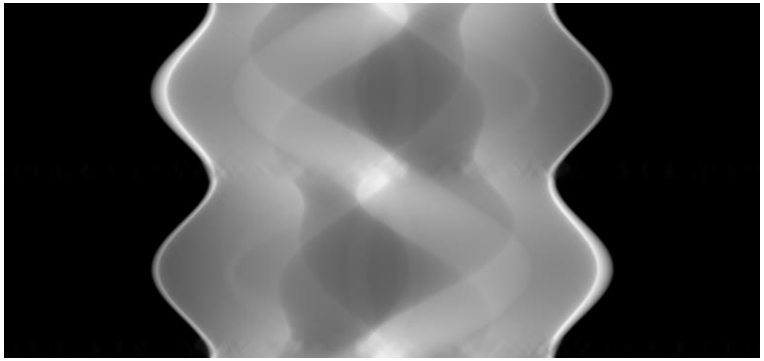
1. Estimation of moment curves

- Linear correlation coefficients, 720 orders



2. Restored sinogram

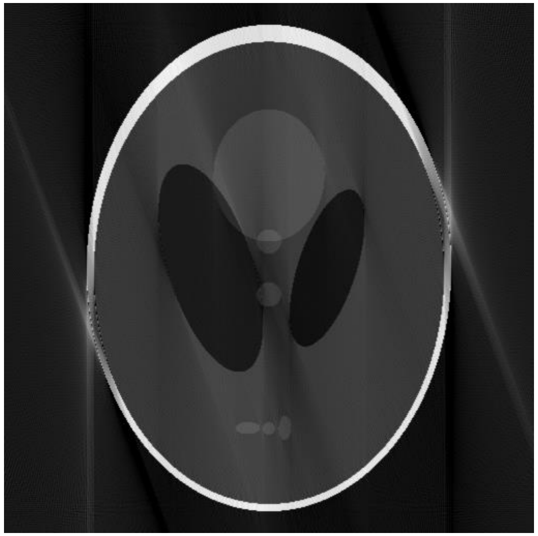
Limited angle sinogram, 160º



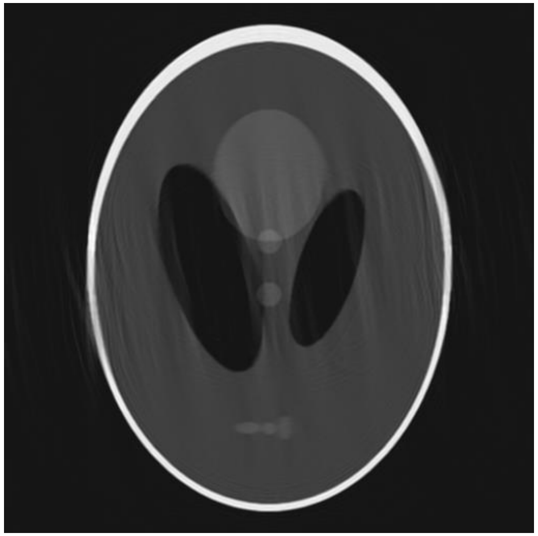
Restored sinogram, 360º

3. Reconstructed images

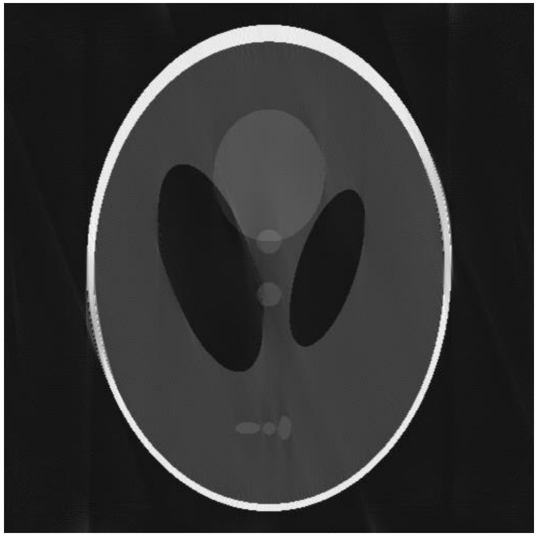
In the final fused image, streaks are reduced, only minor streaks remain, without reintroducing regression artifacts



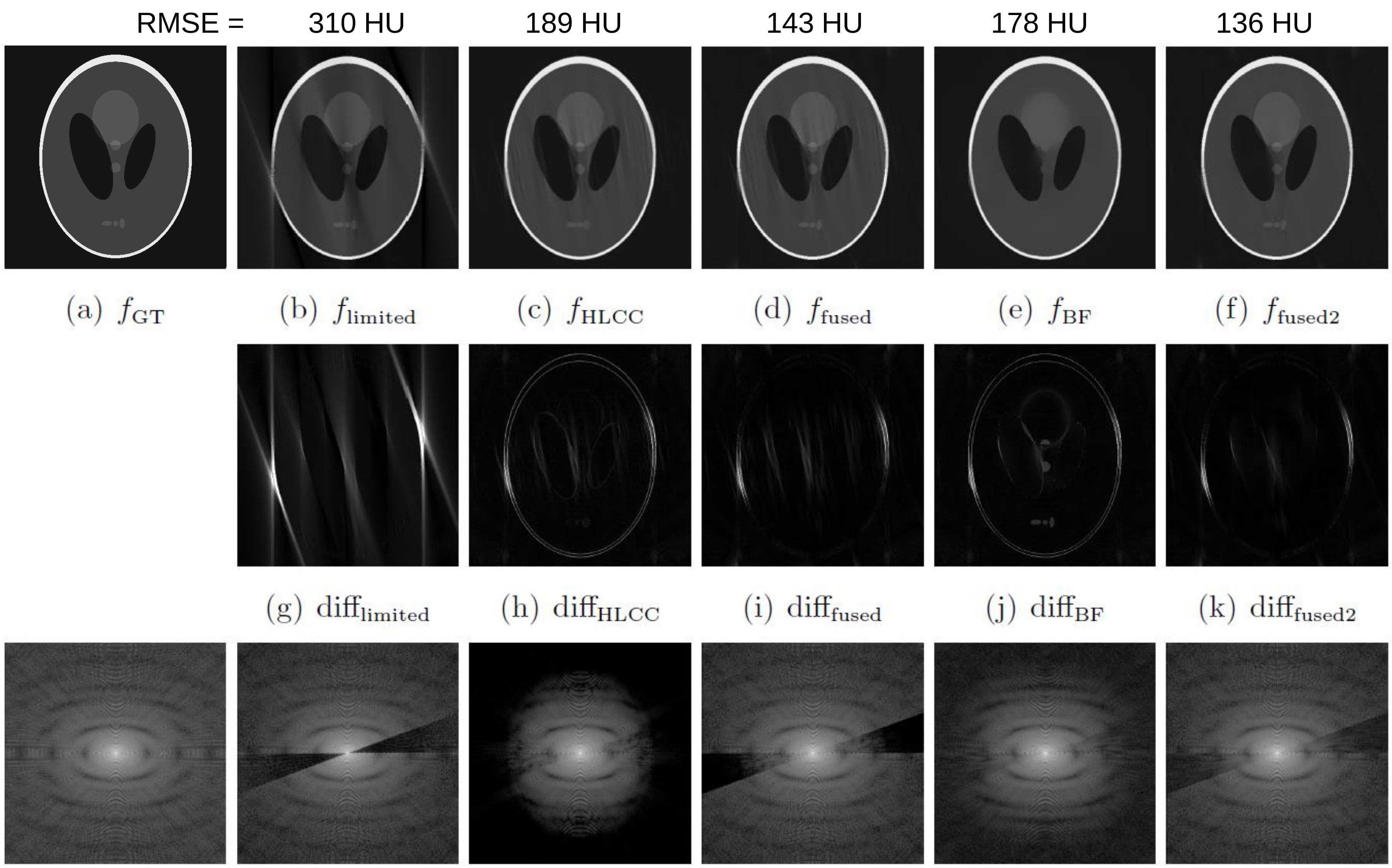
Limited angle reconstruction, $f_{limited}$



Recondtrution of restored sinogram, f_{HLCC}



Fusion with bilateral filter f_{fused2}



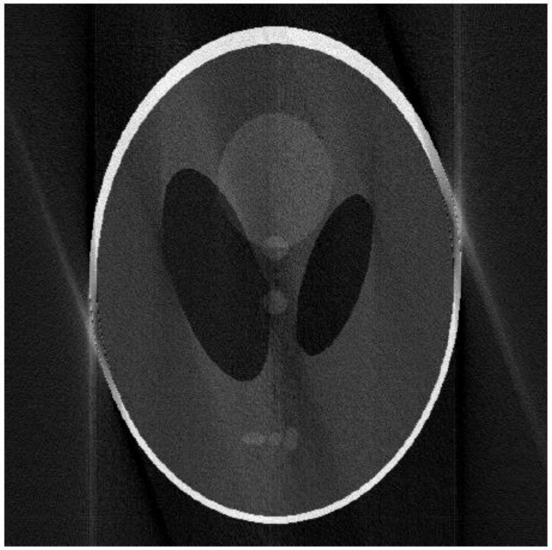
RMSE = 310 HU 189 HU 143 HU 178 HU 136 HU
(a) f_{GT}
(b) f_{limited}
(c) f_{HLCC}
(d) f_{fused}
(e) f_{BF}
(f) f_{fused2}
(g) $\mathrm{diff}_{\mathrm{limited}}$
(h) $\mathrm{diff}_{\mathrm{HLCC}}$
(i) $\mathrm{diff}_{\mathrm{fused}}$
(j) $\mathrm{diff}_{\mathrm{BF}}$
(k) $\mathrm{diff}_{\mathrm{fused2}}$

b. Shepp-Logan phantom, Poisson noise

Reconstructed images, Poisson noise

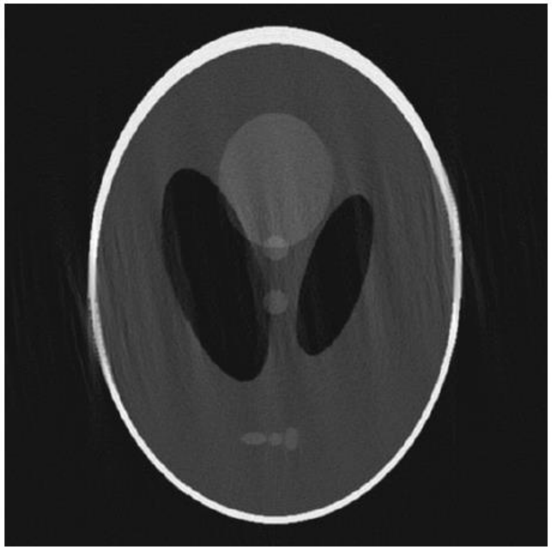
- Poisson noise is suppressed in f_{HLCC}
- Streaks are reduced, although Poisson noise is back in f_{fused2}

RMSE = 361 HU



Limited angle reconstruction, $f_{limited}$

184 HU



Recondtrution of restored sinogram, f_{HLCC}

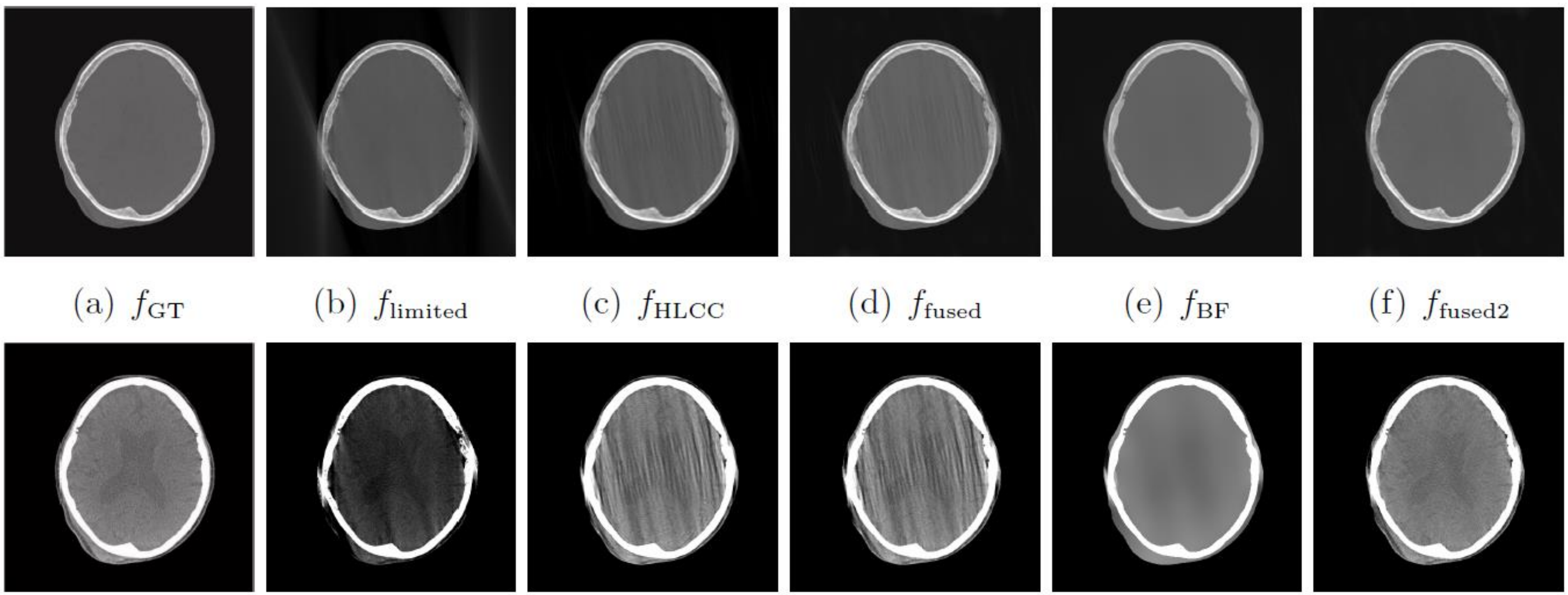
210 HU

Fusion with bilateral filter f_{fused2}

3. Preliminary experiment on clinical data

Reconstructed images

RMSE = 310 HU 189 HU 143 HU 178 HU 136 HU



(a) f_{GT} (b) f_{limited} (c) f_{HLCC} (d) f_{fused} (e) f_{BF} (f) f_{fused2}

(g) f_{GT} (h) f_{limited} (i) f_{HLCC} (j) f_{fused} (k) f_{BF} (l) f_{fused2}

Top row window: [-1200, 2000] HU, bottom row window: [-200, 300] HU

IV. Conclusion

Conclusion

- We propose a regression and fusion method to restore missing data in limited angle tomography
- The regression formulation allows convenient ill-posedness analysis.
- Only low frequency components are correctly restored in the regression step
- Bilateral filtering is utilized to reduce artifacts caused by regression errors and partially recover high frequency components
- We use the restored frequency components to fill in the missing double wedge region only
- Streak artifacts are reduced and intensity offset is corrected without reintroducing artifacts in the final fused image

Thank you for your attention!

Questions and suggestions?