# Restoration of Missing Data in Limited Angle Tomography Based on Consistency Conditions

Yixing Huang[1], Oliver Taubmann[1,3], Xiaolin Huang[1,4], Joachim Hornegger[1,3], Guenter Lauritsch[2], Andreas Maier[1,3]

[1] Pattern Recognition Lab, Department of Computer Science, Friedrich-Alexander University Erlangen-Nuremberg, Erlangen, Germany
[2] Siemens Healthcare GmbH, Forchheim, Germany
[3] Erlangen Graduate School in Advanced Optical Technologies (SAOT), Friedrich-Alexander University Erlangen-Nuremberg, Erlangen, Germany
[4] Institute of Image Processing and Pattern Recognition, Shanghai Jiao Tong University, Shanghai, China

## Introduction

**Limited angle tomography (parallel-beam case)**

- **Definition:** Scan angle $\theta_{\max} < \pi$, here $[\theta_{\min}, \theta_{\max}] = [0°, 160°]$
- **Challenge:** Data incompleteness causing streak artifacts
- **Technique:** Data restoration based on consistency conditions

## Materials and Methods

### I. Background theory

**1. Helgason-Ludwig Consistency Conditions (HLCC):**

$$a_n(\theta) = \int_{-1}^{1} p(s,\theta) U_n(s) \mathrm{d}s, \qquad b_{n,m} = \frac{1}{2\pi}\int_0^{2\pi} a_n(\theta) e^{-im\theta} \mathrm{d}\theta,$$

where $U_n(s) = \frac{\sin((n+1)\arccos(s))}{\sqrt{1-s^2}}$ and $n$ is the order.

**HLCC**: $b_{n,m} = 0$, when $|m| > n$ or $n + m$ is odd.

**2. Sinogram restoration:**

$$p_{n_r}(s,\theta) = \frac{2}{\pi}\sum_{n=0}^{n_r} a_n(\theta)\big(W(s)\cdot U_n(s)\big),$$

where $n_r$ is the number of orders used and $W(s) = \sqrt{1-s^2}$.

**3. Fourier property of the restored sinogram:**

$W(s)\cdot U_n(s)$ is a high-pass filter (cut-off frequency $w_{c,n} \approx n$).

Only a circular area with radius $w_{c,n_r}$ is restored in the Fourier domain when reconstructing from $p_{n_r}(s,\theta)$.

### II. Proposed method [1]:

**1. Regression method for sinogram restoration [2]:**

- Based on HLCC, when $n$ is even, only even terms,
  $$\underline{[\mathbf{1}, \cos(2\boldsymbol{\theta}), \sin(2\boldsymbol{\theta}), \cos(4\boldsymbol{\theta}), \sin(4\boldsymbol{\theta}), \dots, \cos(n\boldsymbol{\theta}), \sin(n\boldsymbol{\theta})]}\boldsymbol{\beta}_n = a_n(\boldsymbol{\theta});$$
  when $n$ is odd, only odd terms,
  $$\underline{[\cos(\boldsymbol{\theta}), \sin(\boldsymbol{\theta}), \cos(3\boldsymbol{\theta}), \sin(3\boldsymbol{\theta}), \dots, \cos(n\boldsymbol{\theta}), \sin(n\boldsymbol{\theta})]}\boldsymbol{\beta}_n = a_n(\boldsymbol{\theta}),$$
  where $\boldsymbol{\beta}_n$ is the vector of Fourier series coefficients.
- Regression form:
  $$\underline{\boldsymbol{X}_n(\boldsymbol{\theta})}\boldsymbol{\beta}_n = a_n(\boldsymbol{\theta}).$$
- Ill-conditioned, **Lasso regression**:
  $$\boldsymbol{\beta}_n = \operatorname{argmin} \frac{1}{2}\|\boldsymbol{X}_n(\boldsymbol{\theta})\boldsymbol{\beta}_n - a_n(\boldsymbol{\theta})\| + \tau_n\|\boldsymbol{\beta}_n\|_1,$$
  where $\tau_n = 0.001\cdot\left(1 - \frac{n}{1000}\right)$.

**2. Bilateral filtering (BF):**

- Reconstruction from restored sinogram, denoted by $f_{\mathrm{HLCC}}$;
- Apply BF to $f_{\mathrm{HLCC}}$, get $f_{\mathrm{BF}}$ and its Fourier transform $F_{\mathrm{BF}}$.

**3. Image fusion in frequency domain:**

- Reconstruction from limited angle sinogram $f_{\mathrm{limited}}$ and its Fourier transform $F_{\mathrm{limited}}$;
- A binary double wedge mask $M$, $M(w,\theta)|_{\theta_{\max}\le\theta<\pi, -\infty<w<\infty} = 0$;
- Fusion at frequency domain:
  $$F_{\mathrm{fused}}(w,\theta) = F_{\mathrm{limited}}(w,\theta)\cdot M(w,\theta) + F_{\mathrm{BF}}(w,\theta)\cdot\big(1 - M(w,\theta)\big).$$

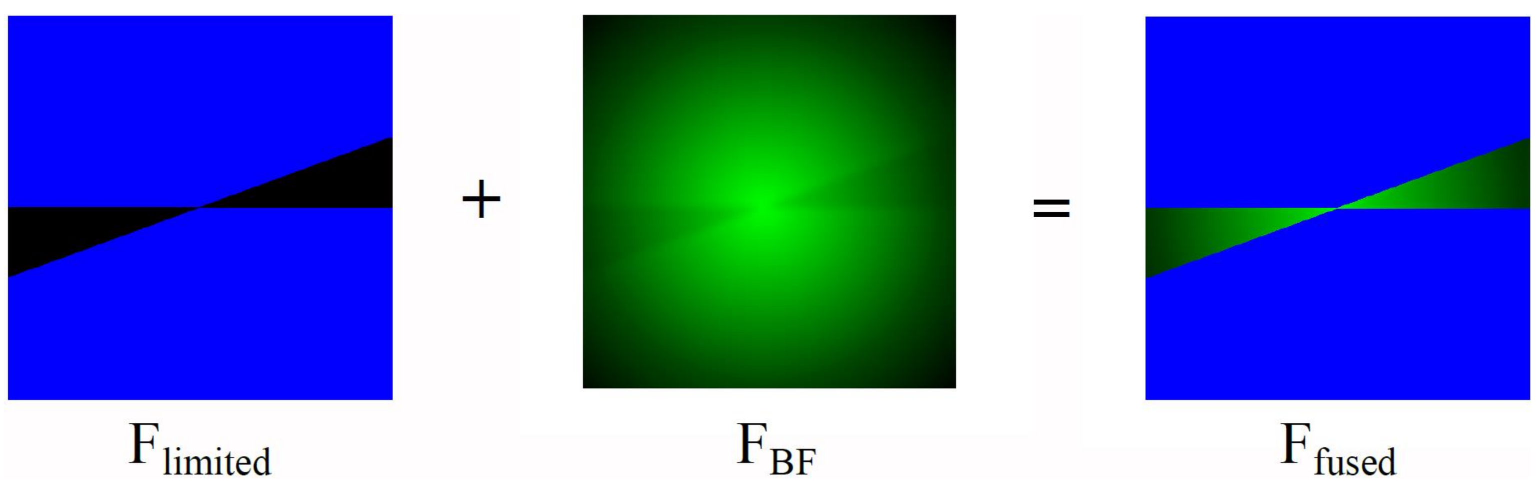



Fig. 1: Illustration of the image fusion in frequency domain. The black, blue, and green areas are the missing, measured, and HLCC estimated frequencies.

## Results

**High contrast Shepp-Logan Phantom study:**

Angular step $0.5^\circ$, $1537$ detector pixels with pixel size $0.2$ mm, image size $512 \times 512$ with pixel size $0.4$ mm

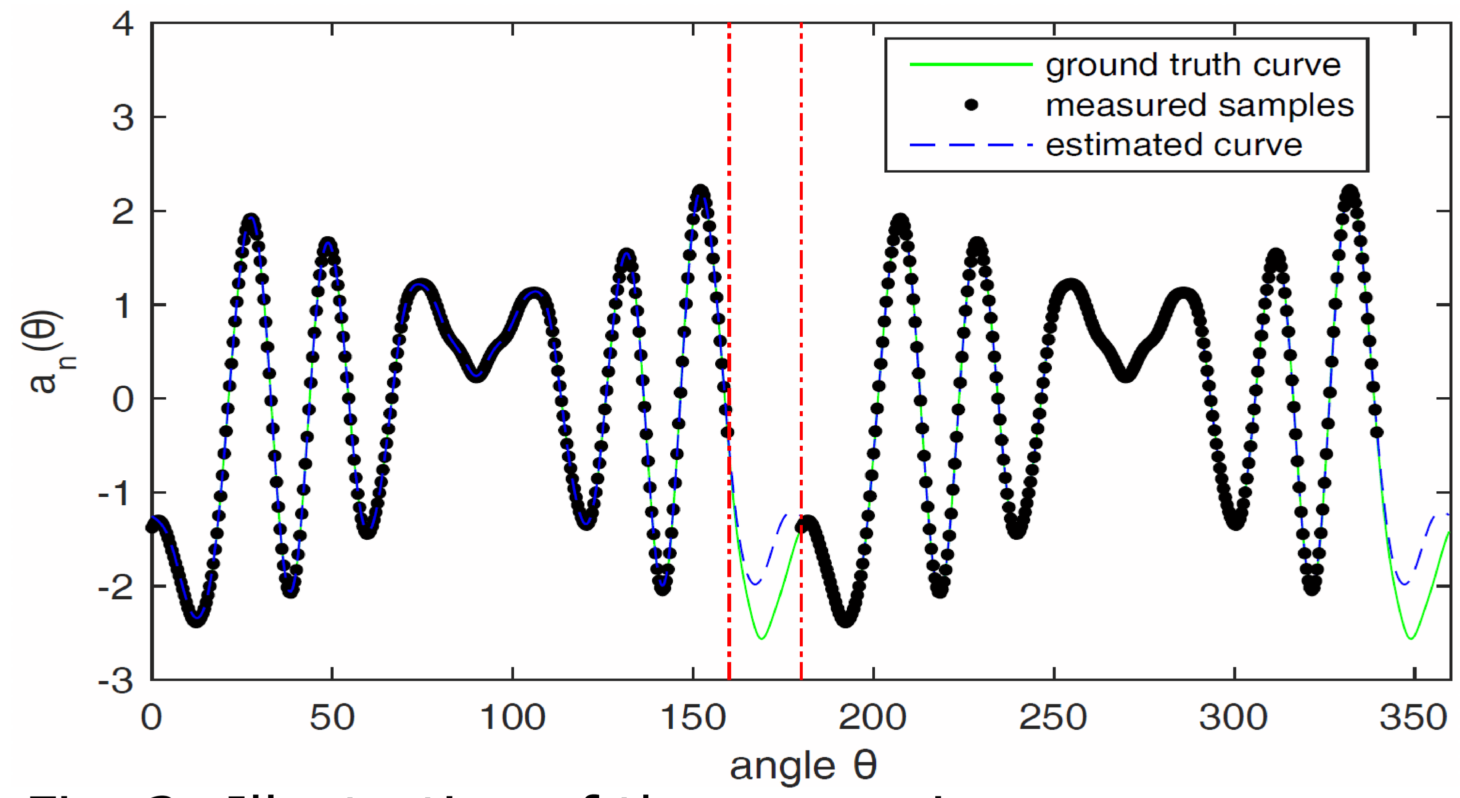
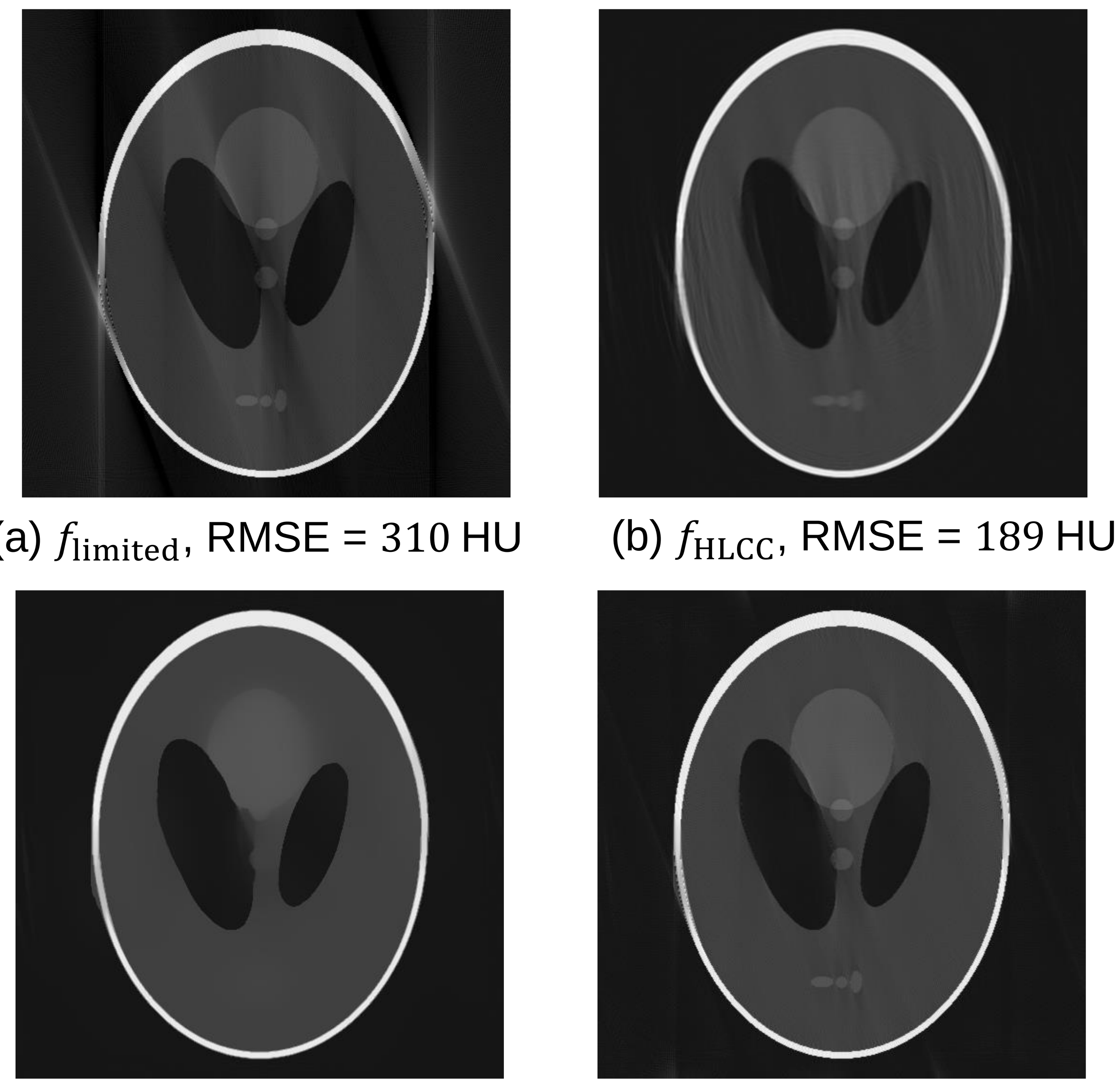



Fig. 2: Illustration of the regression.



(a) $f_{\mathrm{limited}}$, RMSE = 310 HU (b) $f_{\mathrm{HLCC}}$, RMSE = 189 HU

(c) $f_{\mathrm{BF}}$, RMSE = 178 HU (d) $f_{\mathrm{fused}}$, RMSE = 136 HU

Fig. 3: Reconstruction results and their root-mean-square errors (RMSE) w.r.t. the ground truth phantom.

## Discussion and Conclusion

- Missing data restoration in sinogram domain and frequency domain based on HLCC.
- Three techniques: regression, bilateral filtering, and image fusion.
- Low frequency components are restored faithfully.
- Streak artifacts are suppressed in the final image.

## References

[1] Huang, Yixing, et al. "Restoration of Missing Data in Limited Angle Tomography Based on Helgason-Ludwig Consistency Conditions." Biomedical Physics & Engineering Express (2017).
[2] Louis, Alfred K., and W. Törnig. "Picture reconstruction from projections in restricted range." Mathematical Methods in the Applied Sciences 2.2 (1980).

**Contact**
✉ yixing.yh.huang@fau.de
www5.cs.fau.de/~yixing