Papoulis-Gerchberg Algorithms for Limited Angle Tomography Using Data Consistency Conditions

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Introduction

Limited Angle Tomography

- C-arm angiographic systems: restrictions from other parts or obstacles
- Image reconstruction from insufficient angular range data
- Artifacts caused by missing data, typically streak artifacts

General Papoulis-Gerchberg (P-G) Algorithm

- Extrapolation of band-limited signals
- Given
  - $g_0(t)$: measured at $[-t_0, t_0]$ while 0 outside this segment
  - $M_g(t)$: a binary mask for the measured segment
  - w_c: the cut-off frequency of the original signal $g(t)$
  - $L(w)$: a perfect low-pass filter which has a cut-off frequency of w_c
- The signal $g(t)$ can be estimated iteratively:

$$g^l(t) = g_0(t) + (1 - M_g(t)) \cdot \mathcal{F}_1^{-1}(L(w) \cdot \mathcal{F}_1(g^{l-1}(t)))$$

  - l: the iteration number
  - $\mathcal{F}_d$ is d-dimensional Fourier transform
  - $\mathcal{F}_d^{-1}$: d-dimensional inverse Fourier transform
  - $g^0(t) = g_0(t)$

Material and Methods

Conventional P-G Algorithm Using Object Support (OS)

- An object $f(x, y)$ has a compact support S: *a priori* knowledge
- $F(w, \theta) = \mathcal{F}_2 f(x, y)$ and $2\pi f(x, y) = \mathcal{F}_2 F(w, \theta)$
- $F(w, \theta)$ is a band-limited signal with band S
- P-G algorithm using OS:

$$F^l(w,\theta) = F_{\text{limited}}(w,\theta) + (1 - M_F(w,\theta)) \cdot \mathcal{F}_2^{-1}(L_S(x,y) \cdot \mathcal{F}_2(F^{l-1}(w,\theta)))$$

  - $F_{\text{limited}}(w,\theta)$: measured frequency components
  - $M_F(w,\theta)$: binary mask for the measured frequency components
  - $L_S(x,y)$: a characteristic function of the support S

P-G Algorithm Using Double-Wedge (DW) Property

- The sinogram $p(s,\theta)$ is band-limited with a DW band:
  - $P(w,k) = \mathcal{F}_2 p(s,\theta)$
  - $P(w,k) = 0$ when $|k/w| > r_d$(distance of the farthest point)
- P-G algorithm using DW:

$$p^l(s,\theta) = p_{\text{limited}}(s,\theta) + (1 - M_p(s,\theta)) \cdot \mathcal{F}_2^{-1}(L_{\text{DW}}(w,k) \cdot \mathcal{F}_2(p^{l-1}(s,\theta)))$$

  - $p_{\text{limited}}(s,\theta)$: the measured sinogram
  - $M_p(s,\theta)$: binary mask for the measured frequency components
  - $L_{\text{DW}}(w,k)$: a characteristic function of the DW region

P-G Algorithm Using HLCC

- Helgason-Ludwig consistency conditions (HLCC)
  - $U_n(s)$: the Chebyshev polynomial of the second kind
  - n-th order moment curve: $a_n(\theta) = \int_{-\infty}^{\infty} p(s,\theta) U_n(s)\,\mathrm{d}s$
  - Inverse transform: $p_{n_r}(s,\theta) = \frac{2}{\pi}\sum_{n=0}^{n_r} a_n(\theta)(W(s) \cdot U_n(s))$
  - $b_n(m) = \boldsymbol{F}_1 a_n(\theta)$
  - HLCC: $b_n(m) = 0$, when $|m| > n$ or $m + n$ is odd
- The moment curves $a_n(\theta)$ are band-limited
- P-G algorithm using HLCC:

$$a_n^l(\theta) = a_{n,\text{limited}}(\theta) + (1 - M_{a_n}(\theta)) \cdot \mathcal{F}_1^{-1}(L_{\text{HLCC}}(n,m) \cdot \mathcal{F}_1(a_n^{l-1}(\theta)))$$

  - $a_{n,\text{limited}}(\theta)$: the measured moment curve
  - $M_{a_n}(\theta)$: binary mask for the measured moment curve
  - $L_{\text{HLCC}}(n,m)$: a characteristic function of the HLCC

Material and Methods

P-G Algorithm Using HLCC and Soft-Thresholding (ST)

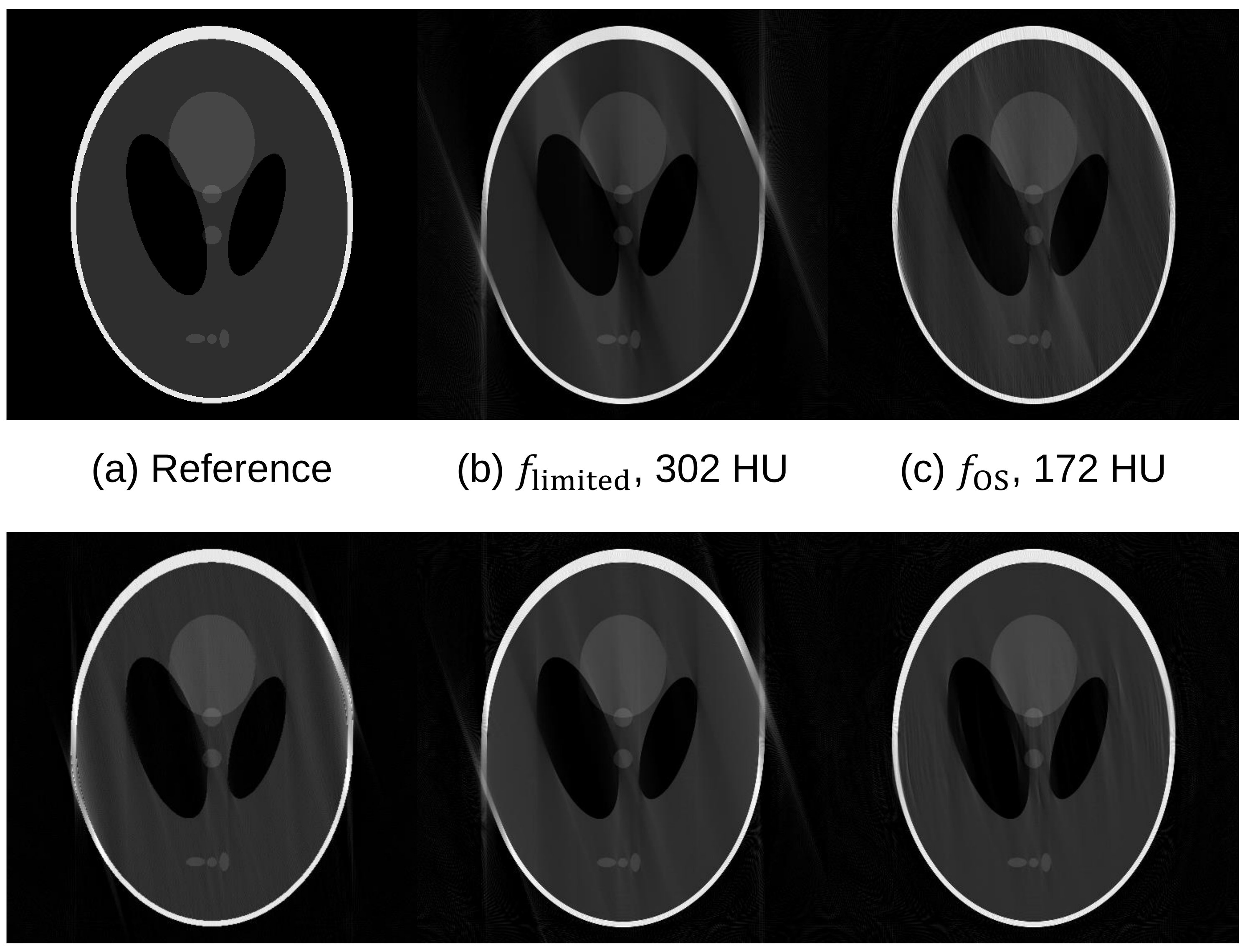
- The Fourier coefficients of moment curves are sparse
- P-G algorithm using HLCC and ST:

$$a_n^l(\theta) = a_{n,\text{limited}}(\theta) + (1 - M_{a_n}(\theta)) \cdot \mathcal{F}_1^{-1}(\mathcal{S}_\tau(L_{\text{HLCC}}(n,m) \cdot \mathcal{F}_1(a_n^{l-1}(\theta))))$$

  - $\mathcal{S}_\tau(v)$: a ST operator with threshold τ

Results and Discussion

- f_{limited}: FBP reconstruction using Ram-Lak filter
- f_{OS}: the OS of the ground truth phantom is assumed to be known
- f_{DW}: $r_p =$ 94 mm, the top point of the phantom
- $f_{\text{HLCC,ST}}$: the threshold is set to $\tau = 0.5\,(1 - n/2500)$
- $f_{\text{HLCC,ST}}$ has the best image quality: the boundary is well restored and streak artifacts inside are also mostly reduced



(a) Reference (b) f_{limited}, 302 HU (c) f_{OS}, 172 HU

(d) f_{DW}, 150 HU (e) f_{HLCC}, 214 HU (f) $f_{\text{HLCC,ST}}$, 75 HU

Fig. 1: Reconstructed images of the Shepp-Logan phantom using different P-G algorithms and their root-mean-square error, parallel-beam, 160° angular range.

Conclusion

- The P-G algorithm using HLCC and ST has the best performance
- It uses the sparsity information of the Fourier coefficients of moment curves

Disclaimer

The concepts and information presented in this paper are based on research and are not commercially available.

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