DMIP – Exercise

Sinograms and Filtered Backprojection (FBP) for Fan Beam

Marco Bögel, Yan Xia
Pattern Recognition Lab (CS 5)

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Fan-Beam Reconstruction

- Start with parallel FBP equation:

$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} p(s,\Theta)\, h(x\cos(\Theta) + y\sin(\Theta) - s) ds d\Theta$$

- Use following identities:

$$\begin{aligned} x &= r\cos(\varphi) \\ y &= r\sin(\varphi) \end{aligned}$$

$$r(\cos(\varphi) cos(\Theta) + \sin(\varphi)\sin(\Theta)) = r\cos(\Theta - \varphi)$$

- This gets us the polar-coordinate representation:

$$f(r,\varphi) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} p(s,\Theta)\, h(r\cos(\Theta - \varphi) - s) ds d\Theta$$

Fan-Beam Reconstruction

- Change of variables (We want to get rid of Θ and s)

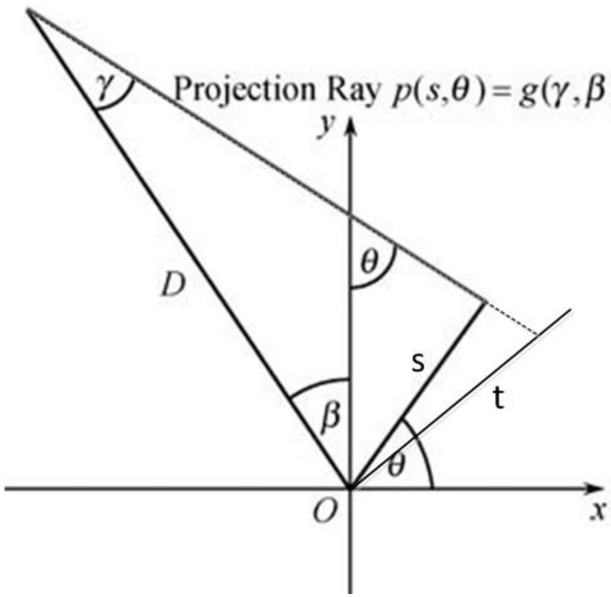
$$f(r,\varphi) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} p(s,\Theta)\, h(r\cos(\Theta-\varphi)-s)dsd\Theta$$

- Use following identities:

$$g(t,\beta) = g(\gamma,\beta) = p(s,\Theta)$$

$$s = \frac{D}{\sqrt{D^2+t^2}}\,t = \cos(\gamma)\,t$$

$$\Theta = \beta + \tan^{-1}(t/D) = \beta + \gamma$$



Fan-Beam Reconstruction

- Change of variables: After some magical math [Zeng09]

$$f(r,\varphi) = \frac{1}{2}\int_0^{2\pi} \frac{1}{U^2} \int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2+t^2}} g(t,\beta)\, h(\hat{t}-t) dt d\beta$$

- With cosine weight:

$$c(t,\beta) = \frac{D}{\sqrt{D^2+t^2}} = \cos(\gamma)$$

- With distance weight:

$$U = \frac{D + r\sin(\beta - \phi)}{D}$$

[Zeng09] Zeng, G. L. (2009). *Medical Image Reconstruction: A Conceptual Tutorial* (p. 198). Berlin, Germany: Springer.

Fan-Beam Backprojection

- Similar approach as in parallel beam, however:
    - Projection of pixels onto detector not parallel
    - Intersection of rays need to be computed for each pixel
    - We have to perform cosine weighting before backprojection
    - During the backprojection we need to apply distance weighting
    - The distance weight depends on the projection angle and pixel position
    - → Distance weight needs to be calculated for each pixel

Fan-Beam Backprojection

- Intersection of two lines (Hesse normal form)

$$\begin{aligned} \text{X-ray line:} \quad \vec{x}^{\mathrm{T}}\,\vec{n}_r - d_r &= 0 \\ \text{Detector line:} \quad \vec{x}^{\mathrm{T}}\,\vec{n}_s - d_s &= 0 \end{aligned}$$

- Reshape to matrix form

$$\begin{pmatrix} \vec{n_{r_1}} & \vec{n_{r_2}} \\ \vec{n_{s_1}} & \vec{n_{s_2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_r \\ d_s \end{pmatrix}$$

- Solve by calculating the inverse

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \vec{n_{r_1}} & \vec{n_{r_2}} \\ \vec{n_{s_1}} & \vec{n_{s_2}} \end{pmatrix}^{-1} \begin{pmatrix} d_r \\ d_s \end{pmatrix}$$