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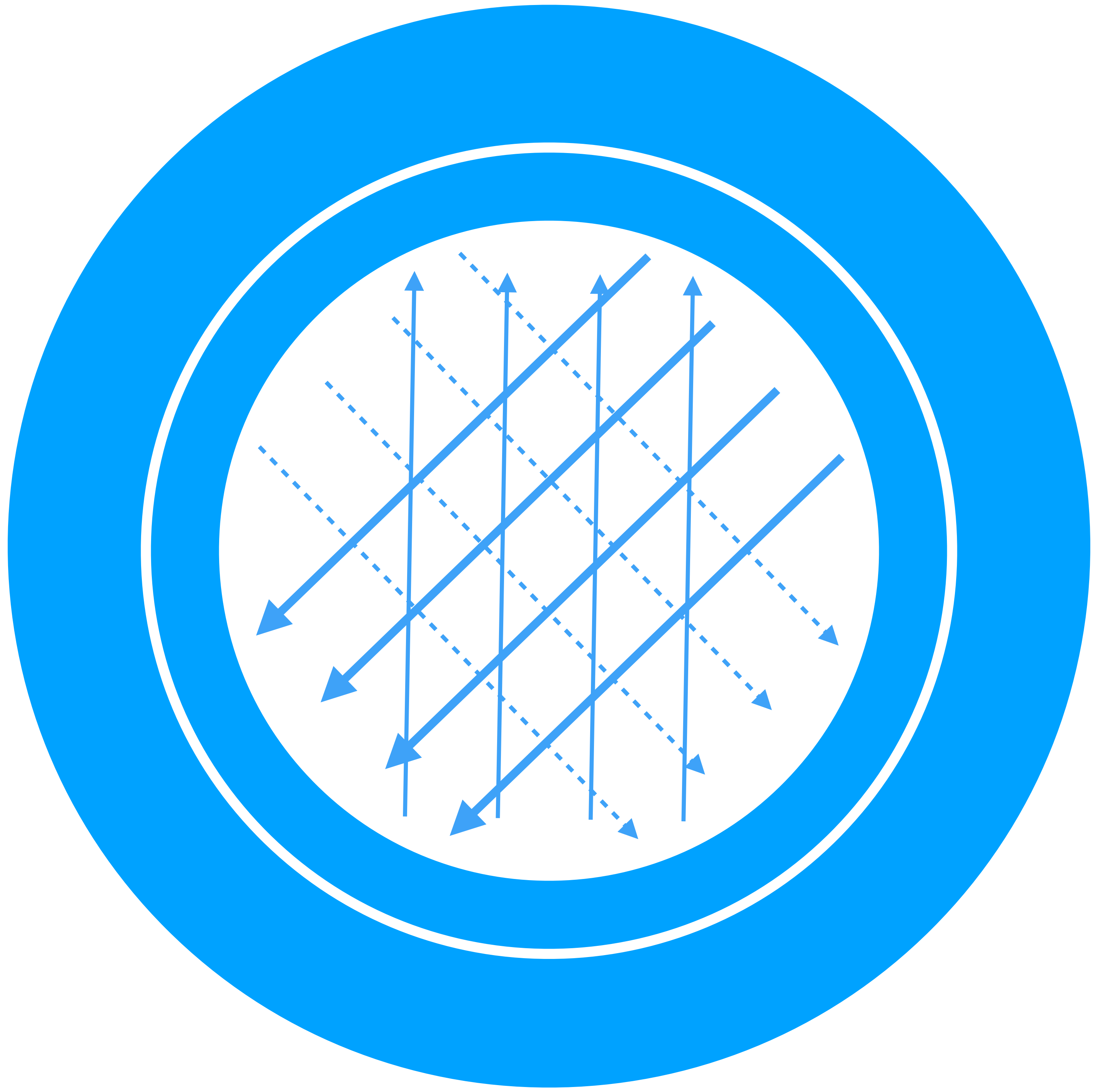
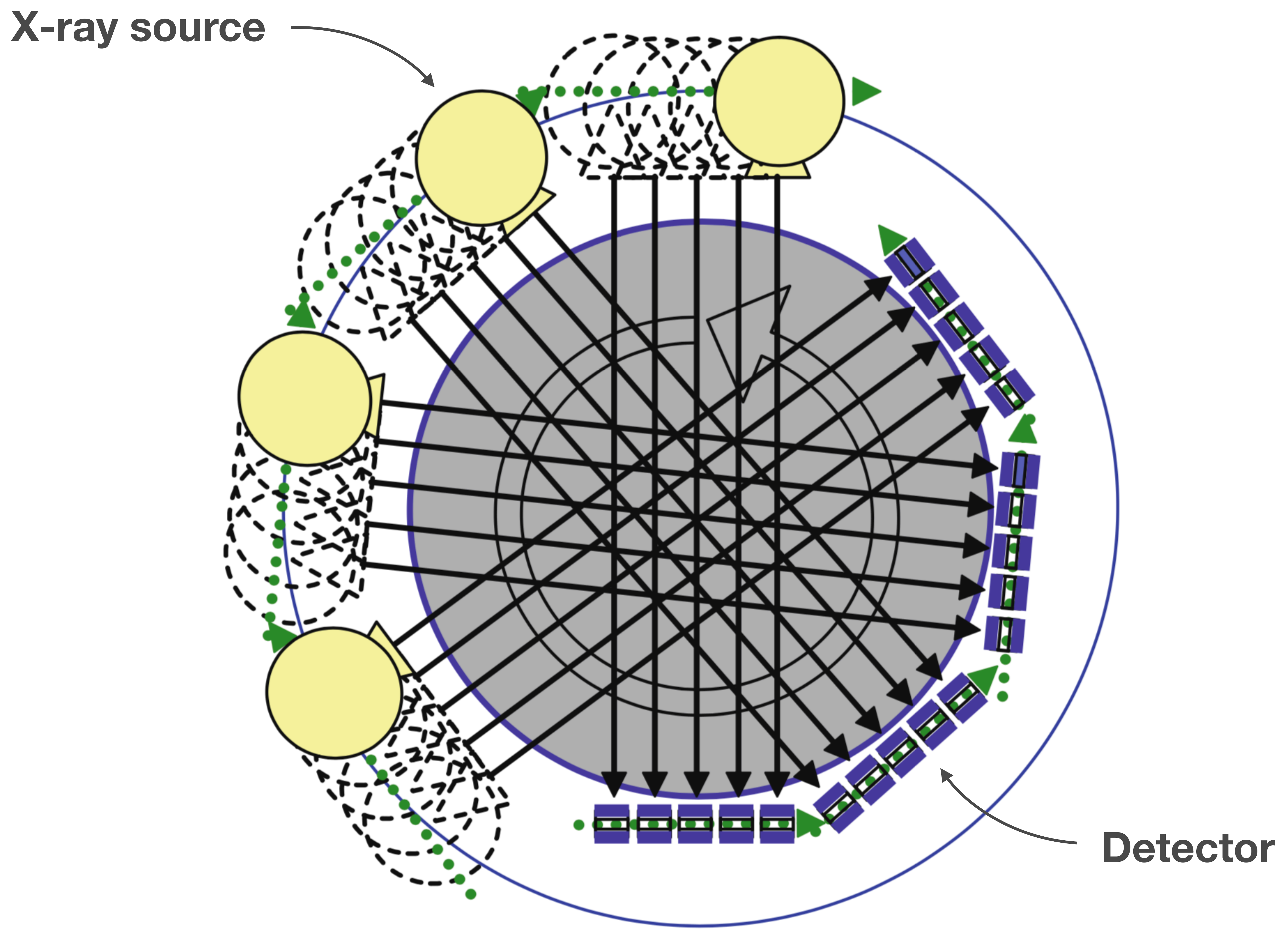
2018-09-24



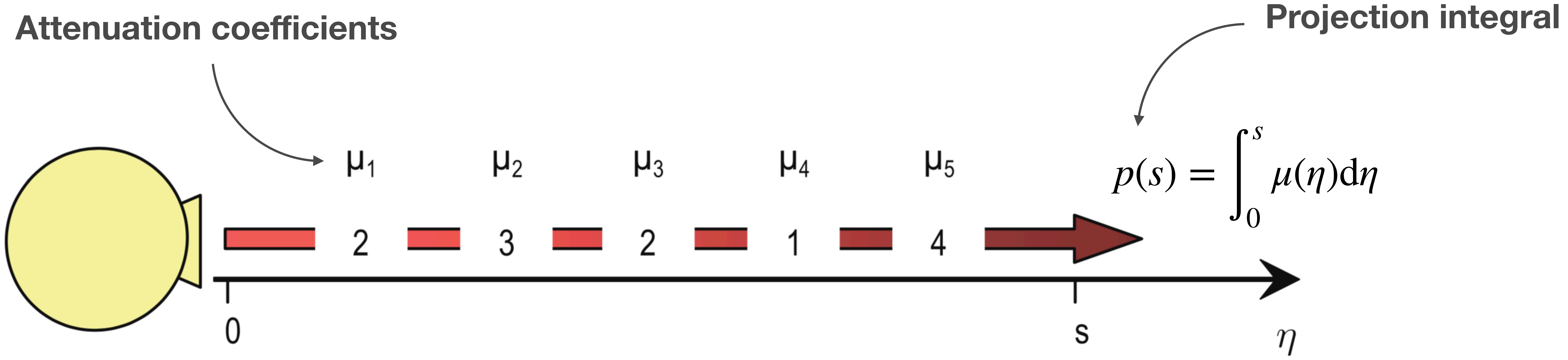
Computed Tomography

Analytical Reconstruction

Introduction (1/2)

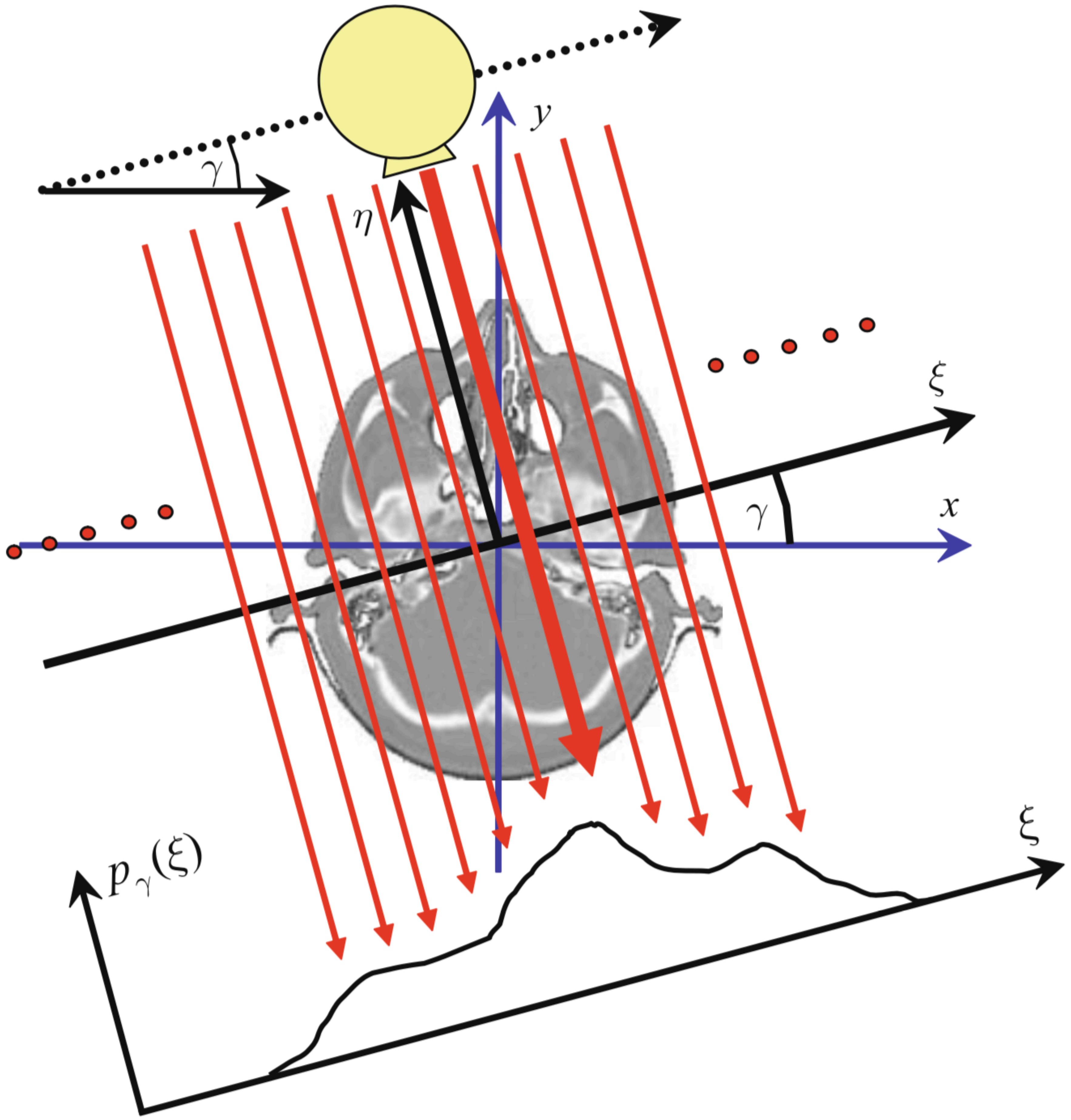


Introduction (2/2)



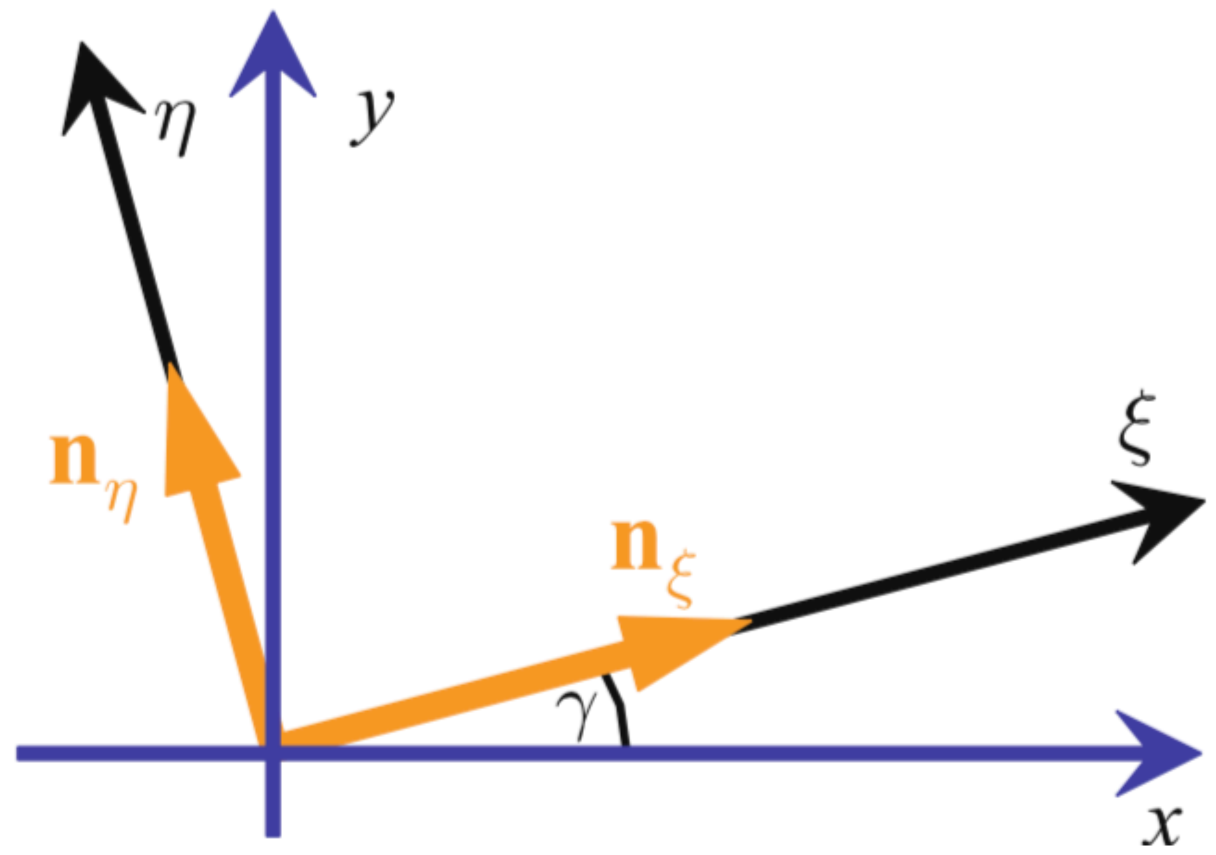
$$p(s) = \int_0^s \mu(\eta)\mathrm{d}\eta$$

Fixed and Rotating Coordinate systems



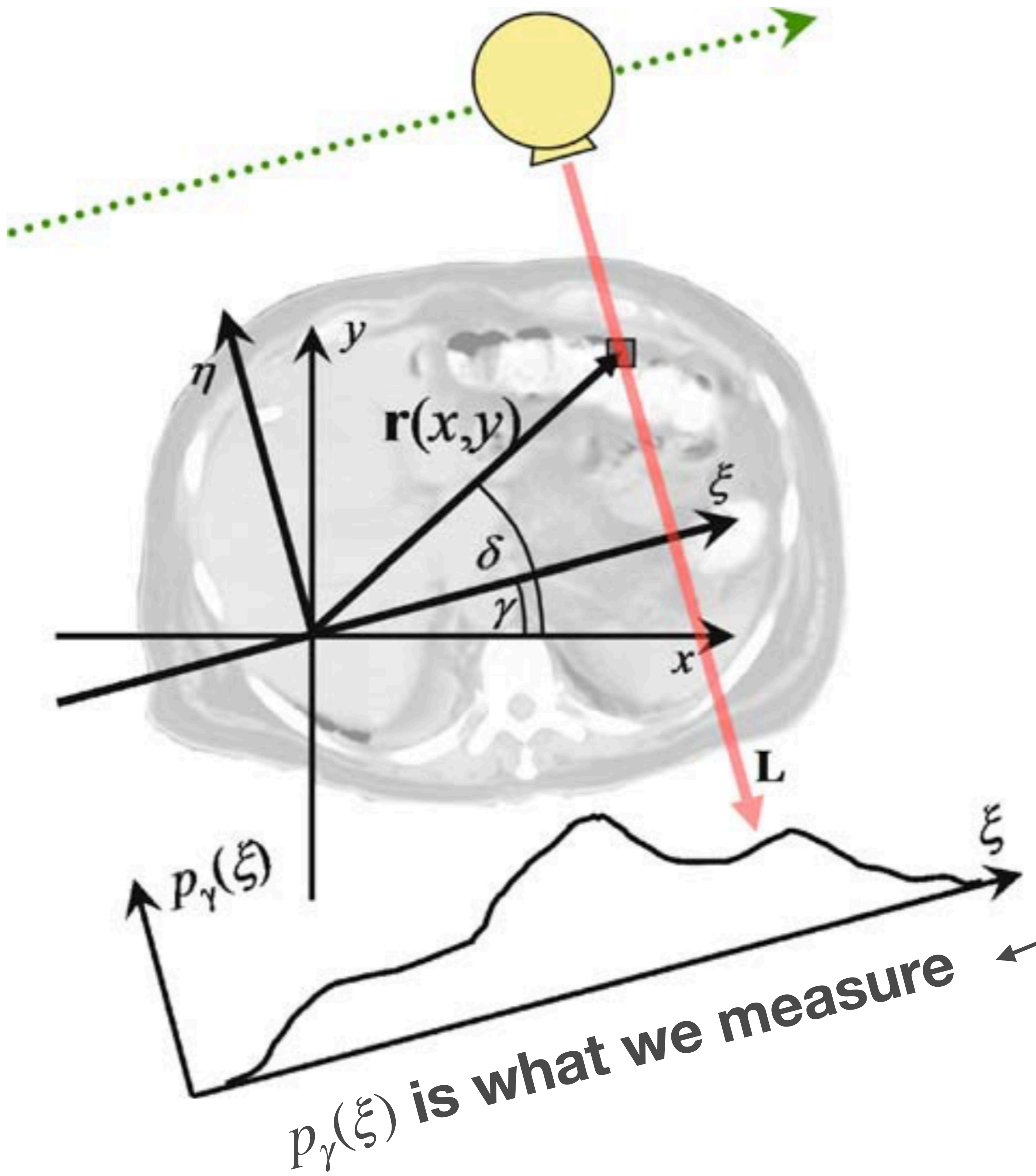
$$\mathbf{n}_\xi = \begin{pmatrix} \cos(\gamma) \\ \sin(\gamma) \end{pmatrix} \qquad \mathbf{n}_\eta = \begin{pmatrix} -\sin(\gamma) \\ cos(\gamma) \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$f(x,y) = \mu\big(\xi(x,y), \eta(x,y)\big) = \mu\big((\mathbf{r}^T \cdot \mathbf{n}_\xi), (\mathbf{r}^T \cdot \mathbf{n}_\eta)\big)$$



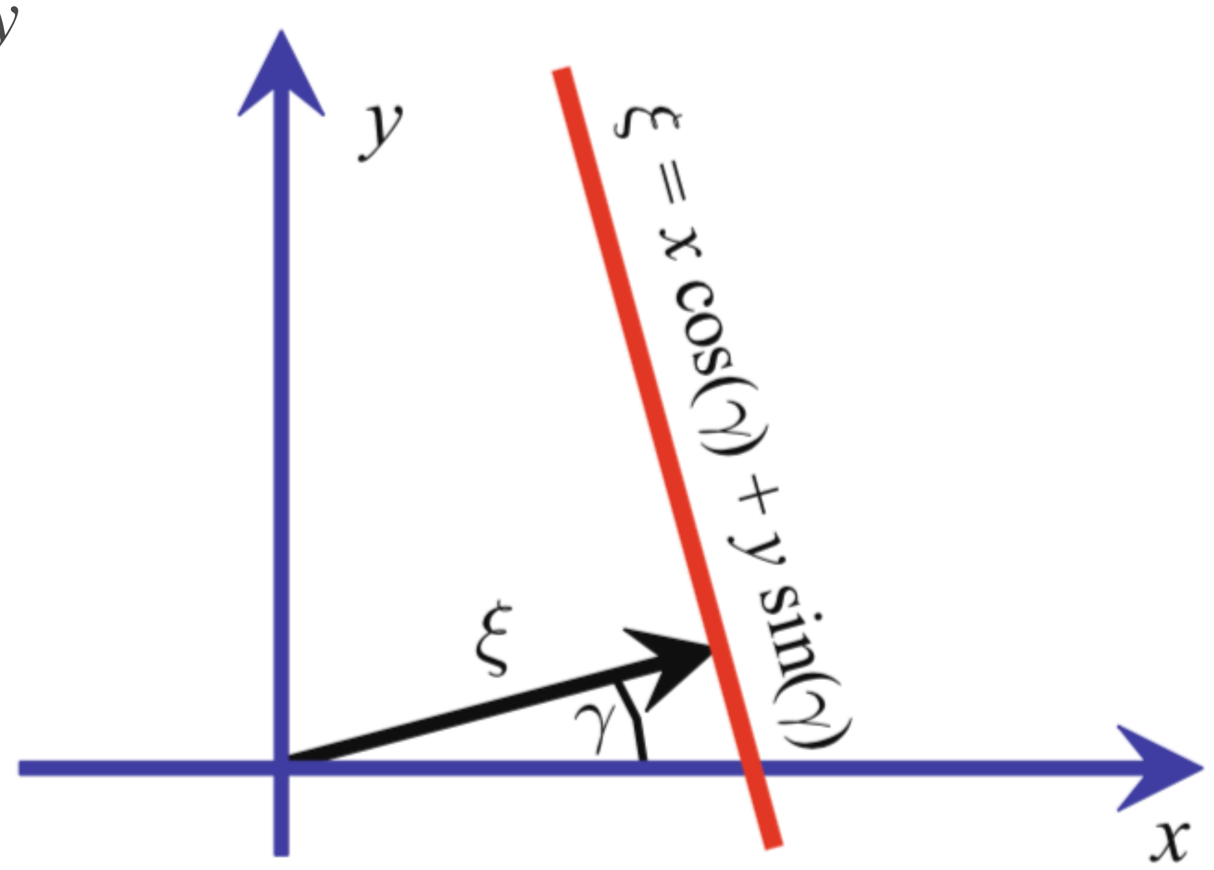
Radon Transform (1/2)

- **Stepwise shift of X-ray source as sampling process of continuous projection signal**
- **L represents path of X-Ray photon and can be described via Hessian normal form using γ and ξ**

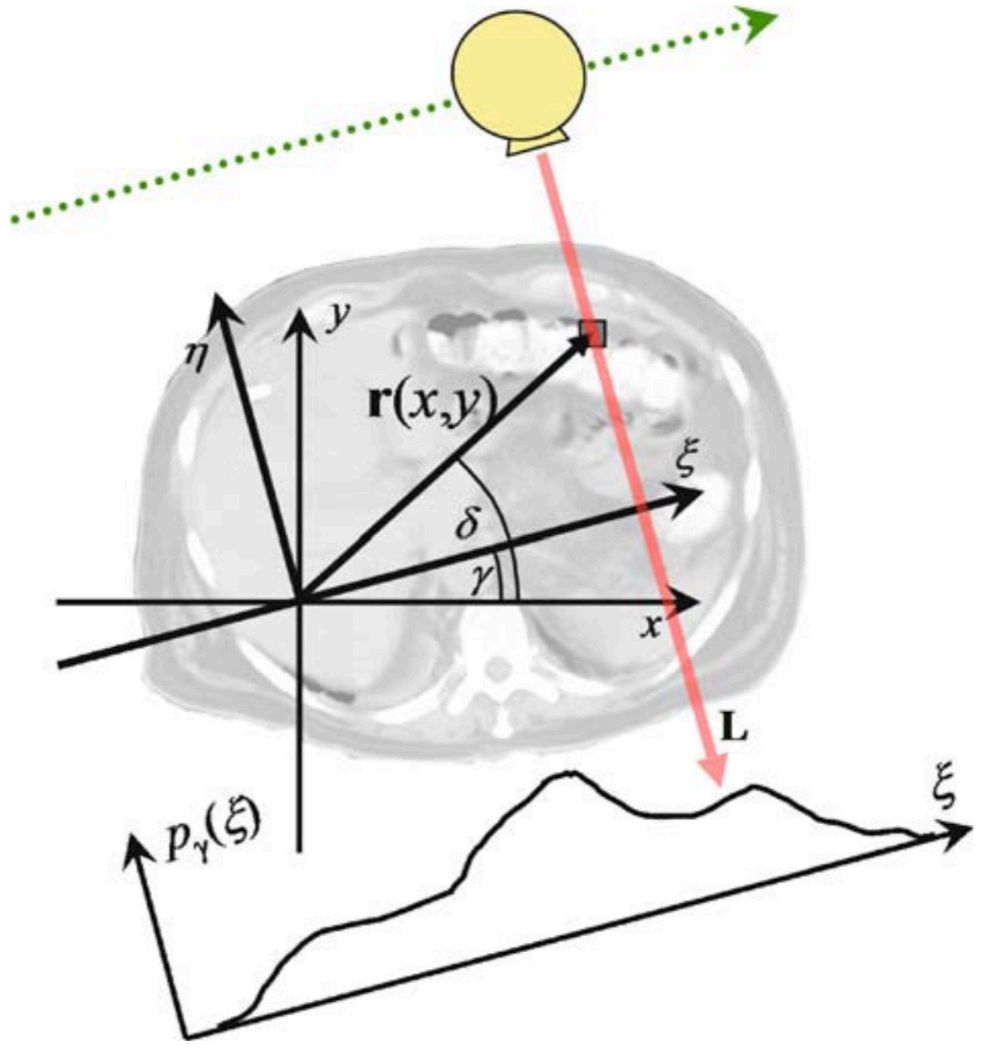


$$f * \delta(\mathbf{L}) = \int_{\mathbb{R}^2} f(\mathbf{r})\delta(\mathbf{r} - \mathbf{L})\mathrm{d}\mathbf{r} = \int_{\mathbf{r}\in\mathbf{L}} f(\mathbf{r})\mathrm{d}\mathbf{r}$$

$$\begin{aligned} f * \delta(\mathbf{L}) &= \int f(\mathbf{r})\delta\left((\mathbf{r}^T \cdot \mathbf{n}_\xi) - \xi\right)\mathrm{d}\mathbf{r} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta\left(x\cos(\gamma) + y\sin(\gamma) - \xi\right)\mathrm{d}x\mathrm{d}y \\ &= p_\gamma(\xi) \end{aligned}$$

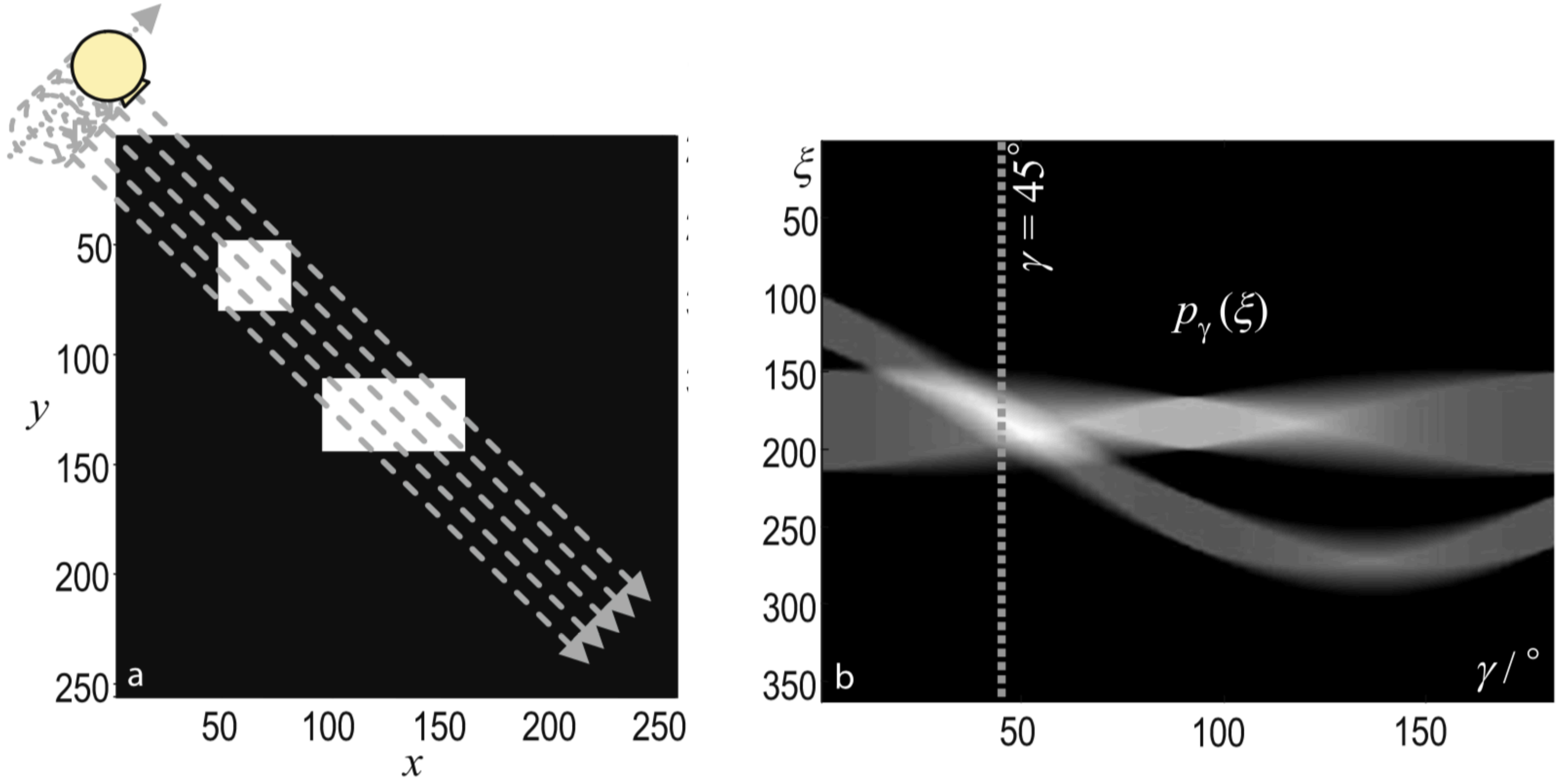


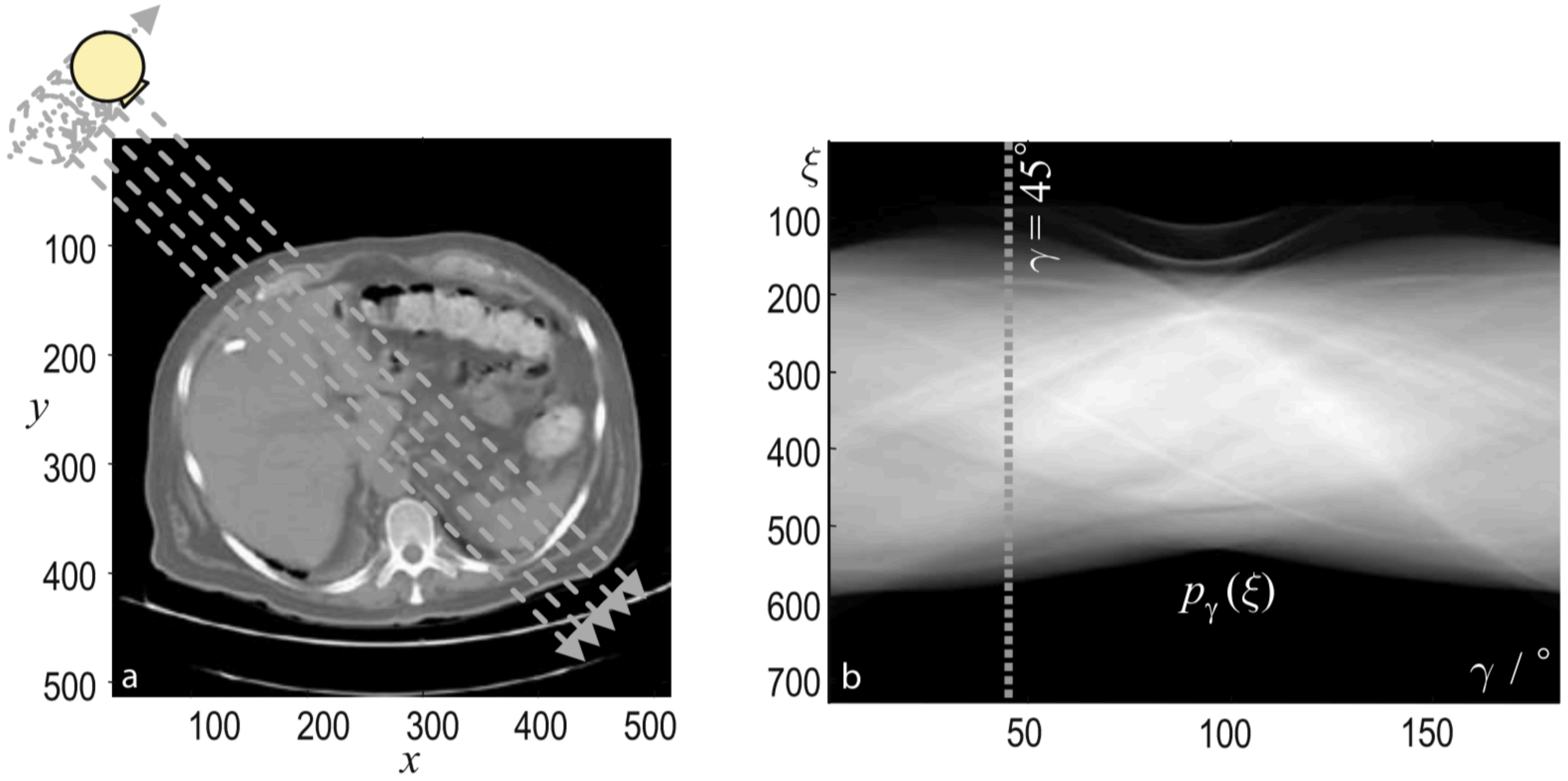
Radon Transform (2/2)



- **Two-dimensional Radon transform:** $p_\gamma(\xi) = \mathscr{R}_2\{f(x,y)\}$

$$
\begin{aligned}
p_\gamma(\xi) &= f * \delta(\mathbf{L}) \\
&= \int f(\mathbf{r})\delta\big((\mathbf{r}^T \cdot \mathbf{n}_\xi) - \xi\big)\mathrm{d}\mathbf{r} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta\big(x\cos(\gamma) + y\sin(\gamma) - \xi\big)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$





Inverse Radon Transform (1/2)

- **Fourier Slice Theorem:** $F\big(q\cos(\gamma), q\sin(\gamma)\big) = P(q,\gamma) = P_\gamma(q)$
- **1D Fourier transform of projection profile corresponds to radial line in Cartesian Fourier space of the object drawn at the angle of the measurement**

$$P_\gamma(q) = \int_{-\infty}^{\infty} p_\gamma(\xi) e^{-2\pi i q \xi} \mathrm{d}\xi = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \mu(\xi,\eta) d\eta \right\} e^{-2\pi i q \xi} \mathrm{d}\xi$$

$$P_\gamma(q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu\big(\xi(x,y), \eta(x,y)\big) e^{-2\pi i q (\mathbf{r}^T \cdot \mathbf{n}_\xi)} \mathrm{d}x\mathrm{d}y$$

$$P_\gamma(q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i q (\mathbf{r}^T \cdot \mathbf{n}_\xi)} \mathrm{d}x\mathrm{d}y$$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i (xu+yv)} \mathrm{d}x\mathrm{d}y$$

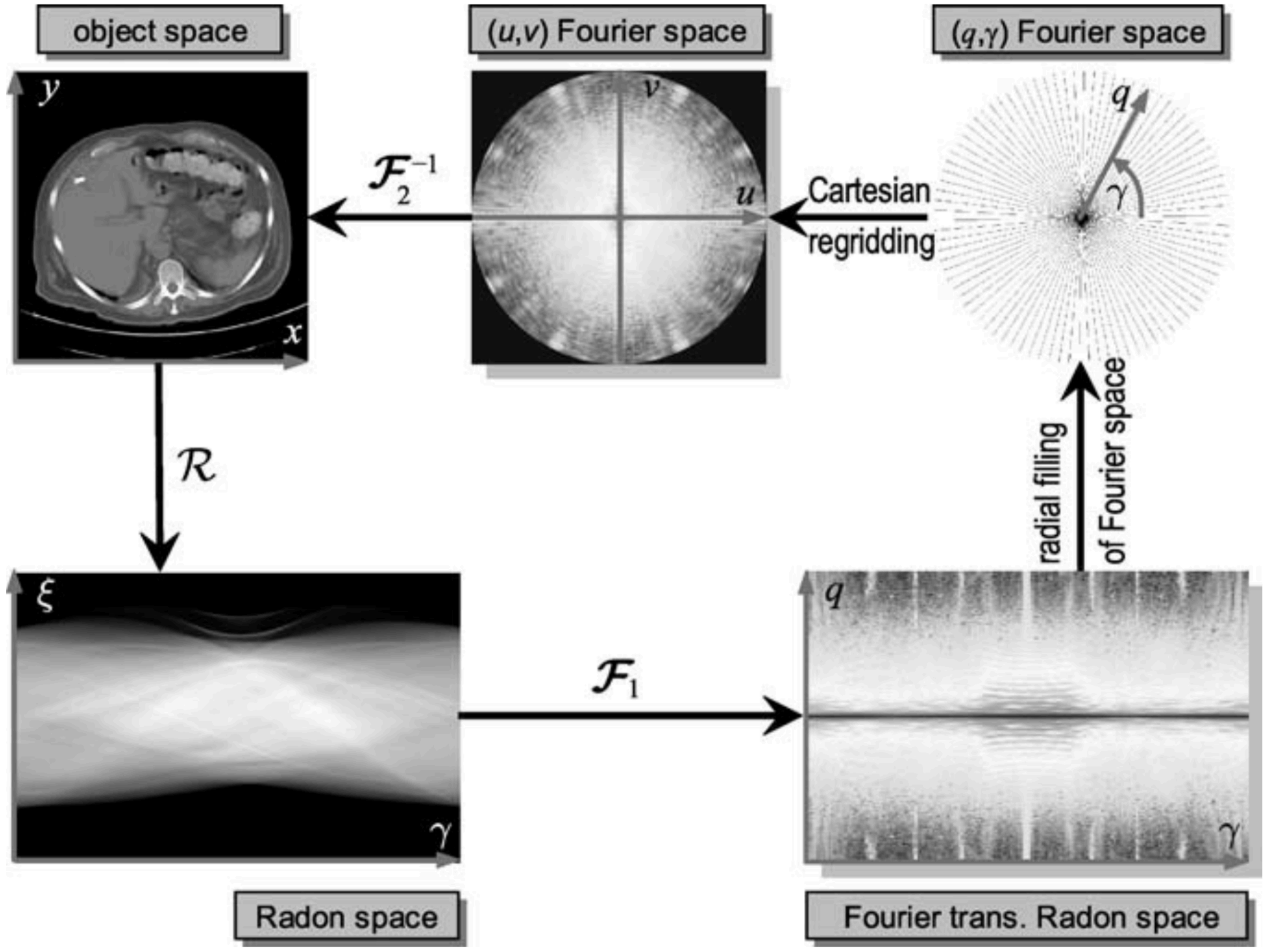
$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i (xq cos(\gamma) + yq\sin(\gamma))} \mathrm{d}x\mathrm{d}y$$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i q (\mathbf{r}^T \mathbf{n}_\xi)} \mathrm{d}x\mathrm{d}y$$

Inverse Radon Transform (2/2)

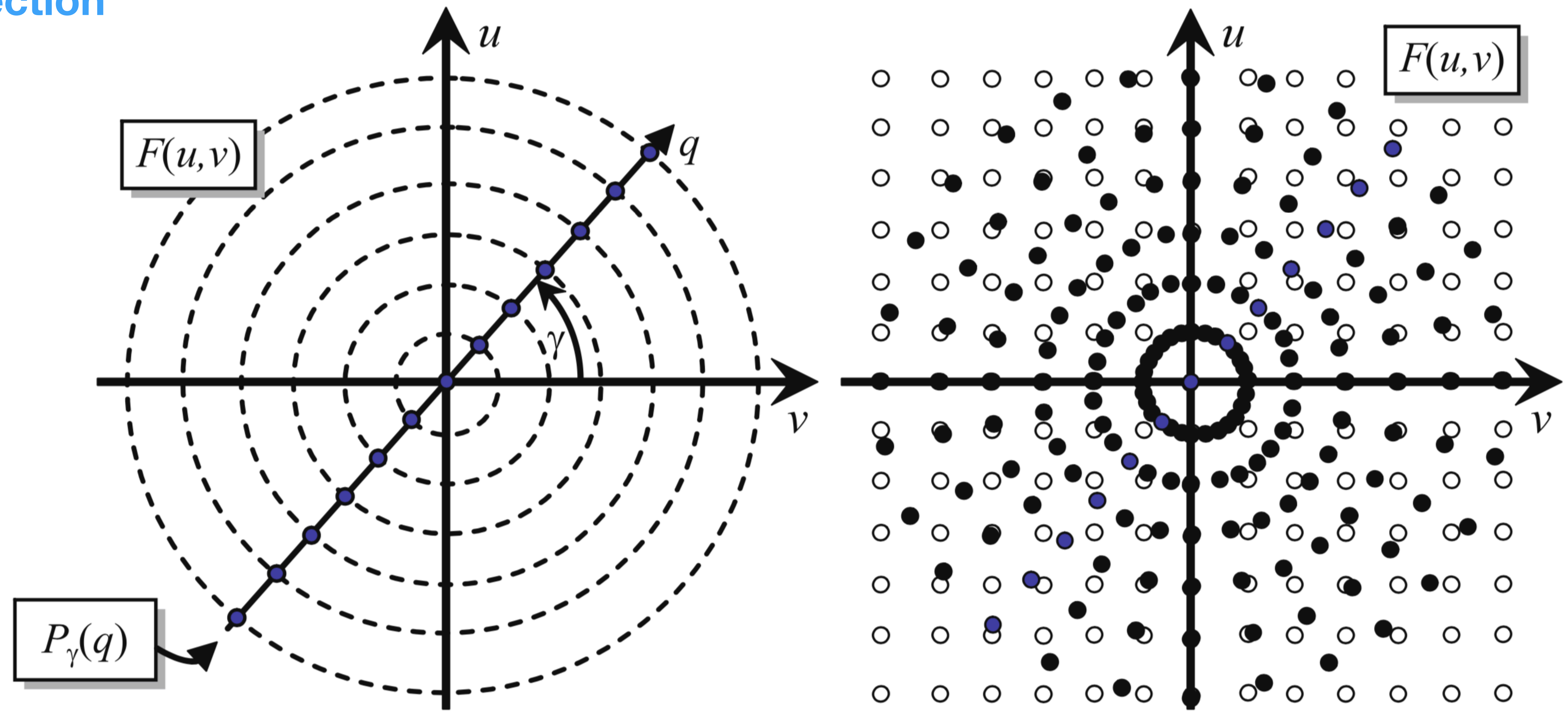
- **Fourier Slice Theorem**: $F\big(q\cos(\gamma), q\sin(\gamma)\big) = P(q,\gamma) = P_\gamma(q)$
- **1D Fourier transform of projection profile corresponds to radial line in Cartesian Fourier space of the object drawn at the angle of the measurement**
- **From Radon space to Object space: 3 easy steps**

Step 1:	$p_\gamma(\xi) \circ\!\!-\!\!\bullet\ P_\gamma(q)$
Step 2:	$P_\gamma(q) \bullet\!\!-\!\!\bullet\ F(u,v)$
Step 3:	$F(u,v) \bullet\!\!-\!\!\circ\ f(x,y)$



Cartesian Regridding

- Cartesian regridding requires interpolation
- Alternatives, e.g.
  - Linogram method
  - Filtered Backprojection



Simple Backprojection (1/2)

- **Core idea: Smear back projection profile values into incident direction**

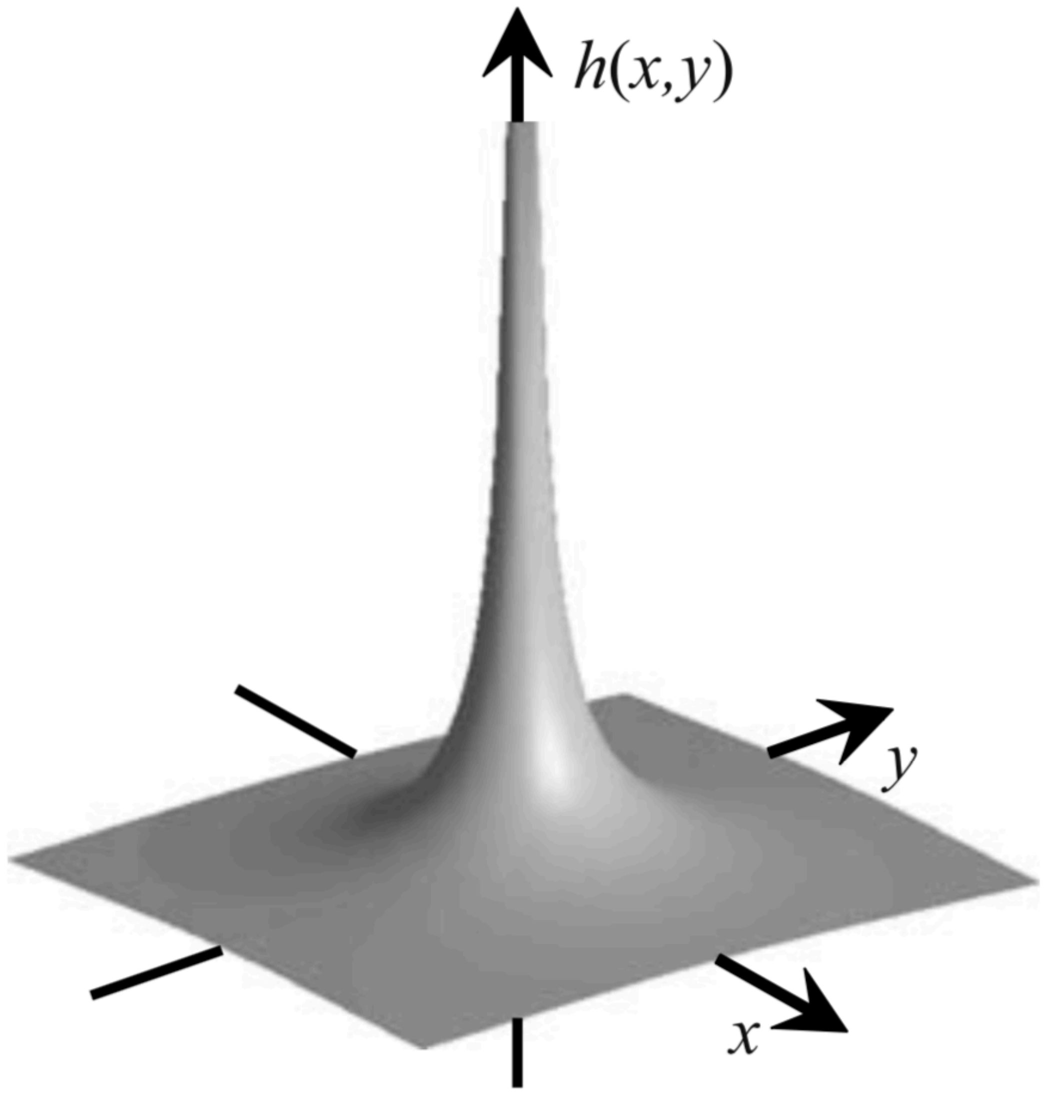
$$g(x, y) = \int_0^{\pi} p_\gamma(\xi) \mathrm{d}\gamma = \int_0^{\pi} p_\gamma\big(x\cos(\gamma) + y\sin(\gamma)\big)\mathrm{d}\gamma$$

- **Problem: projection profile is non-negative**
- **Mathematical insight:**

$$g(x, y) = \int_0^{\pi} \iint_{\mathbf{r}\in\mathcal{R}^2} f(\mathbf{r})\delta(\mathbf{r} - \mathbf{L})\mathrm{d}\mathbf{r}\mathrm{d}\gamma = \cdots = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')\frac{1}{|\mathbf{r} - \mathbf{r}'|}\mathrm{d}\mathbf{r}$$

$$h(x, y) = \frac{1}{|\mathbf{r}|} = \frac{1}{|(x, y)|}$$

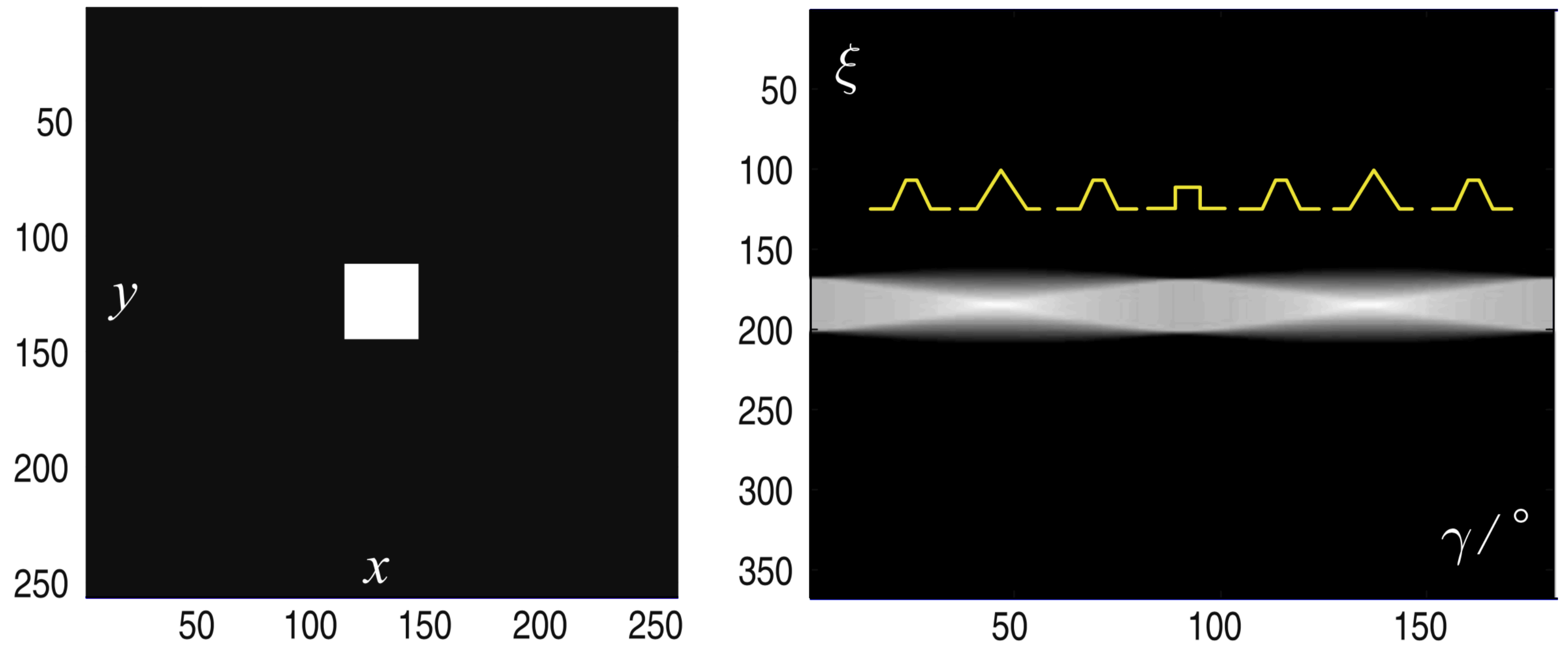
$$g(x, y) = f(x, y) * h(x, y)$$

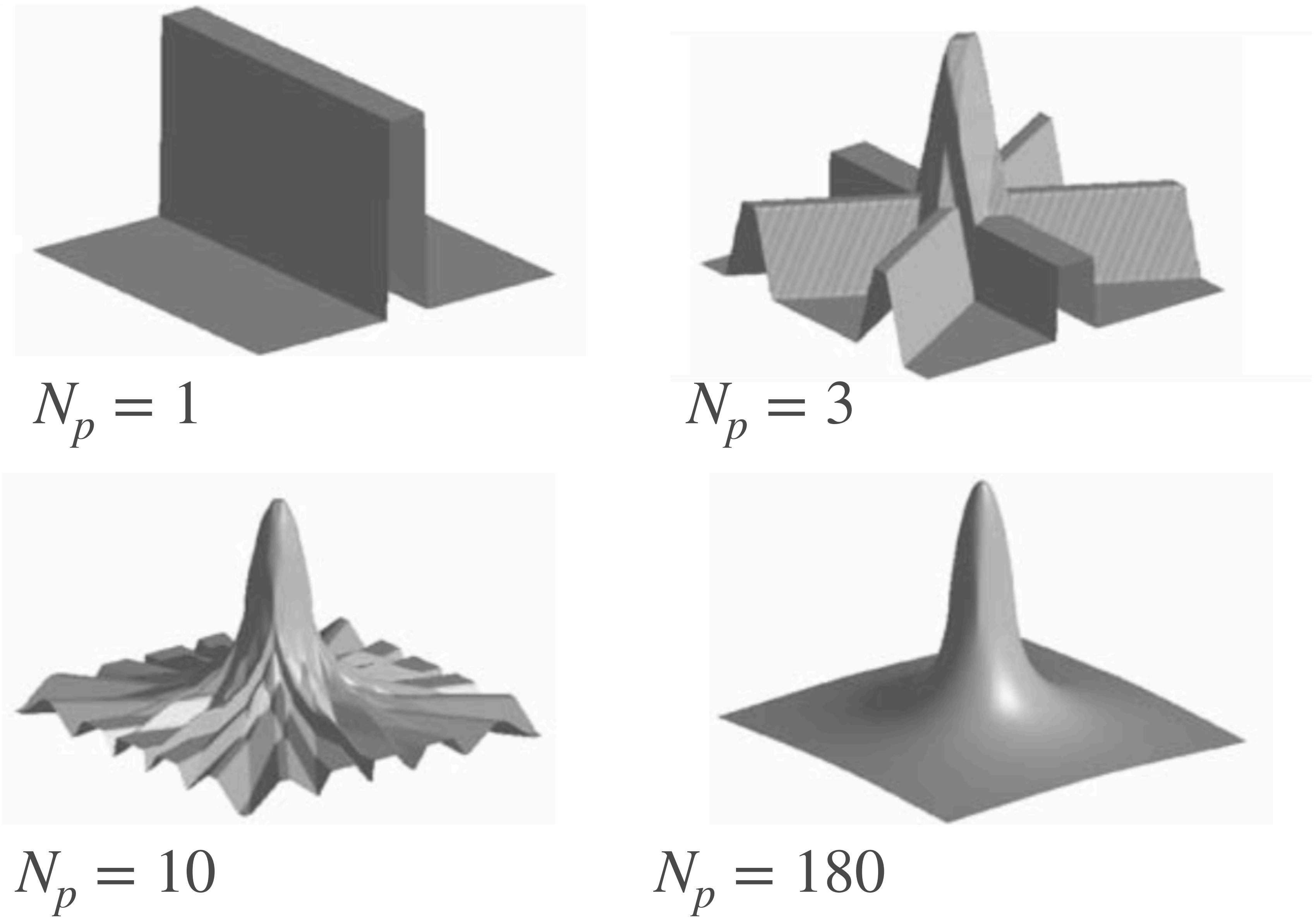


Convolution of original image with filter kernel $h(x, y)$

Simple Backprojection (2/2)

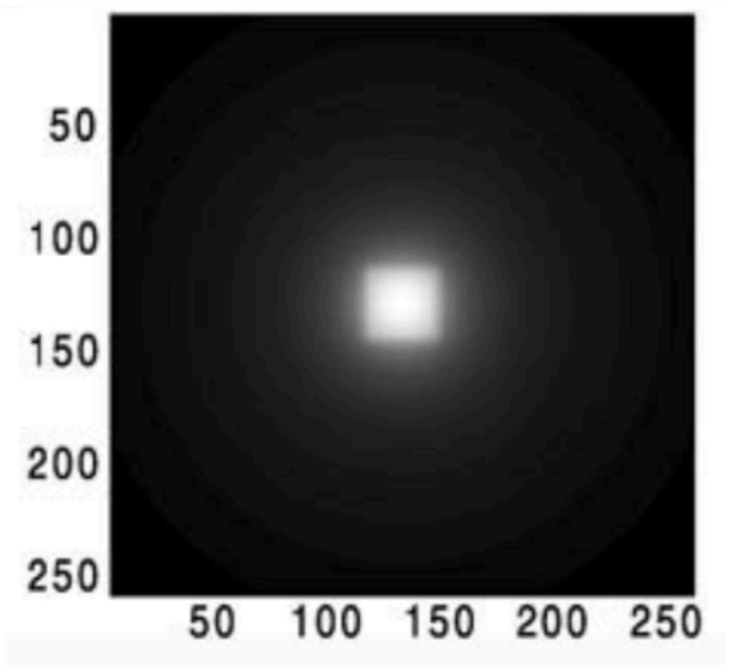
Original image





Number of Projections

Reconstructed image



Filtered Backprojection (1/3)

- **Improved version derived directly from inverse Fourier transform of image**

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{2\pi i(xu+yv)}\mathrm{d}u\mathrm{d}v$$

Express in polar coordinates

$$u = q\cos(\gamma)$$

$$v = q\sin(\gamma)$$

$$J = \det\left(\frac{\partial(u,v)}{\partial(q,\gamma)}\right) = \begin{vmatrix} \frac{\partial u}{\partial q} & \frac{\partial v}{\partial q} \\ \frac{\partial u}{\partial \gamma} & \frac{\partial v}{\partial \gamma} \end{vmatrix} = \begin{vmatrix} \cos(\gamma) & \sin(\gamma) \\ -q\sin(\gamma) & q\cos(\gamma) \end{vmatrix} = q\left(\cos^2(\gamma) + \sin^2(\gamma)\right) = q$$

$$f(x,y) = \int_{0}^{2\pi}\int_{0}^{\infty} F\big(q\cos(\gamma), q\sin(\gamma)\big)e^{2\pi i q(x\cos(\gamma)+y\sin(\gamma))}q\mathrm{d}q\mathrm{d}\gamma$$

$$\vdots$$

$$= \int_{0}^{\pi}\int_{-\infty}^{\infty} F\big(q\cos(\gamma), q\sin(\gamma)\big)e^{2\pi i q(x\cos(\gamma)+y\sin(\gamma))}|q|\mathrm{d}q\mathrm{d}\gamma$$

Exploit symmetry of Fourier transform

Filtered Backprojection (2/3)

- **Reminder: Fourier Slice Theorem:** $F\big(q\cos(\gamma), q\sin(\gamma)\big) = P_\gamma(q)$

$$f(x,y) = \int_0^\pi \int_{-\infty}^{\infty} F\big(q\cos(\gamma), q\sin(\gamma)\big) e^{2\pi i q (x\cos(\gamma) + y\sin(\gamma))} |q| \mathrm{d}q \mathrm{d}\gamma$$

↓

$$f(x,y) = \int_0^\pi \int_{-\infty}^{\infty} P_\gamma(q) e^{2\pi i q \xi} |q| \mathrm{d}q \mathrm{d}\gamma$$

↓

$$h_\gamma(\xi) = \int_{-\infty}^{\infty} P_\gamma(q) |q| e^{2\pi i q \xi} \mathrm{d}q$$

$$f(x,y) = \int_0^\pi h_\gamma(\xi) \mathrm{d}\gamma$$

Backprojection of filtered projection $h_\gamma(\xi)$

Filtered Backprojection (3/3)

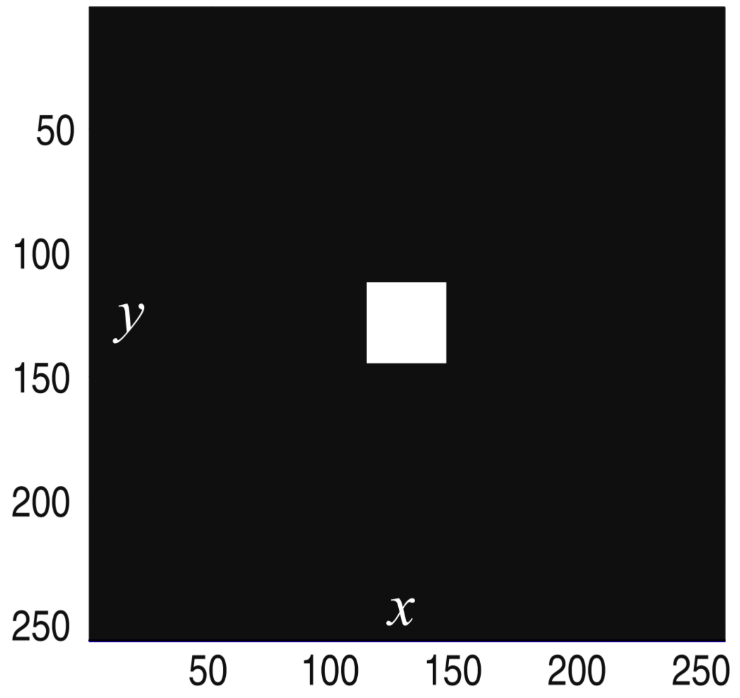
- **From Radon space to Object space: Another 3 easy steps**

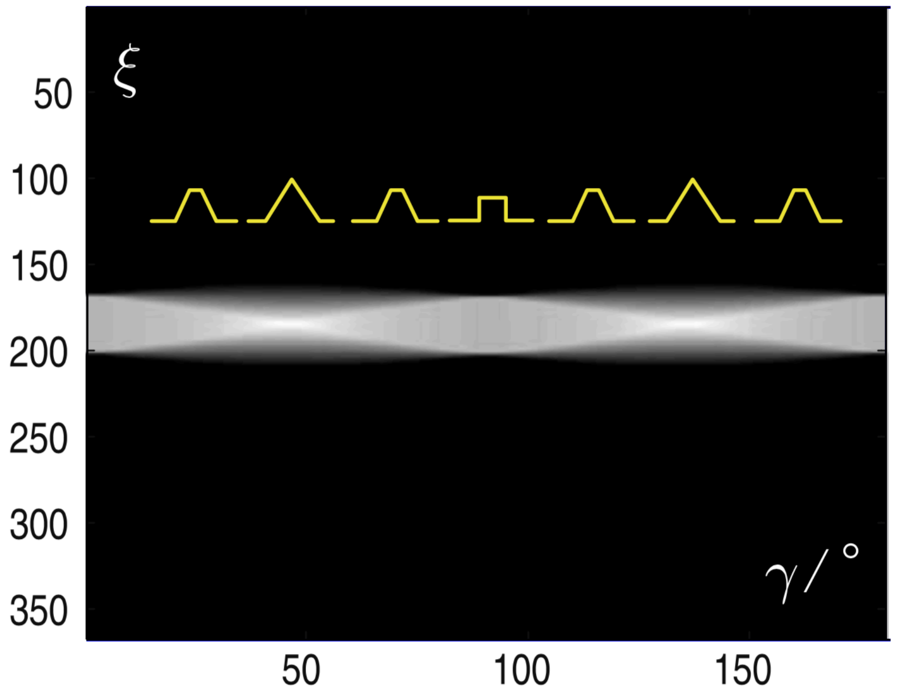
Step 1: $p_\gamma(\xi) \circ\!\!-\!\!\bullet \; P_\gamma(q)$

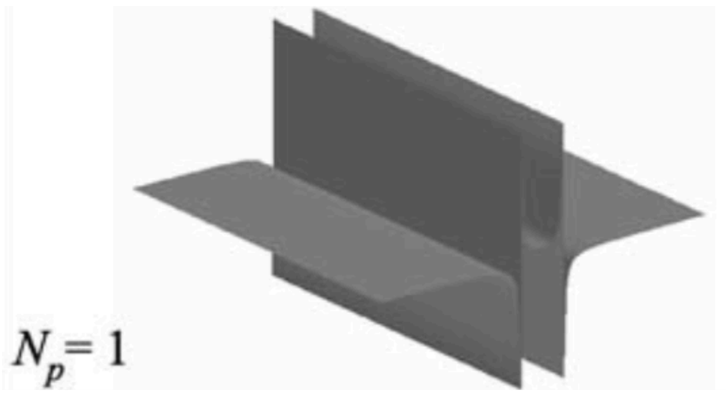
Step 2: $|q|P_\gamma(q) \bullet\!\!-\!\!\circ \; h_\gamma(\xi)$

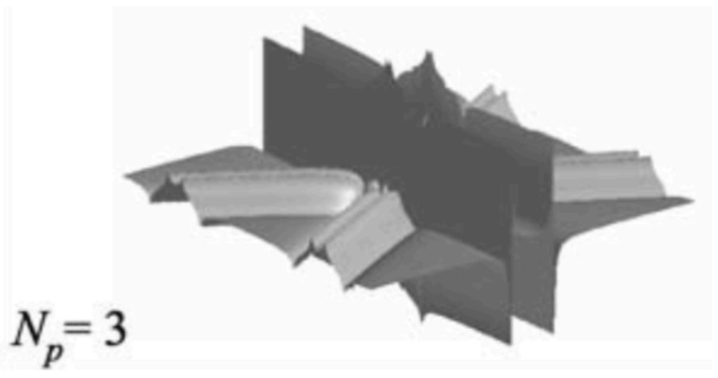
Step 3: $f(x,y) = \int_0^\pi h_\gamma(\xi)\mathrm{d}\gamma$

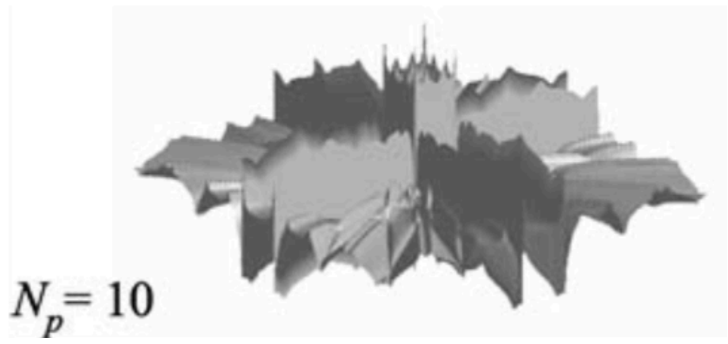
Original image

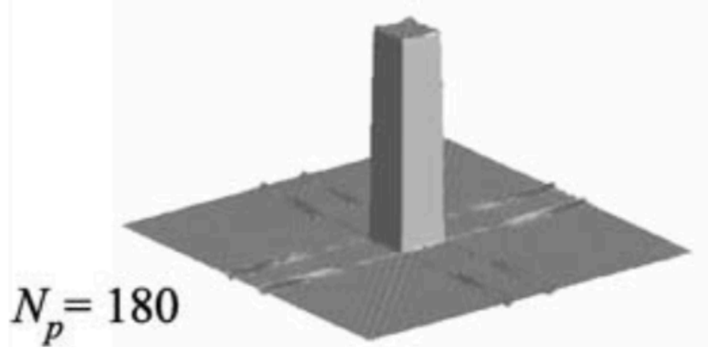




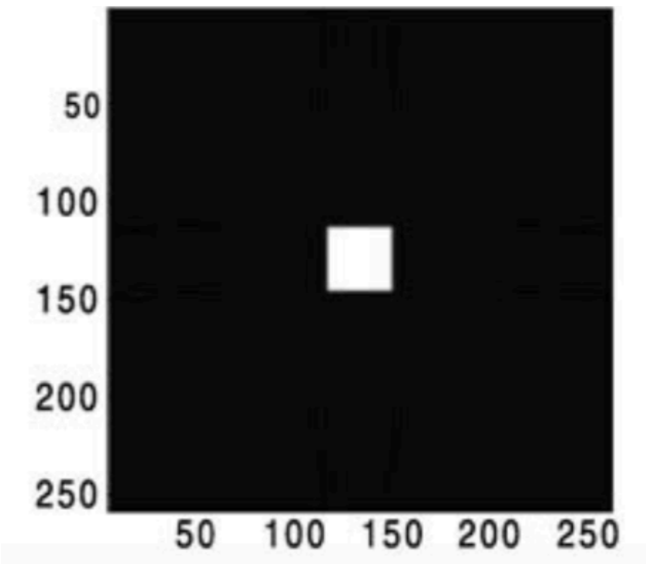






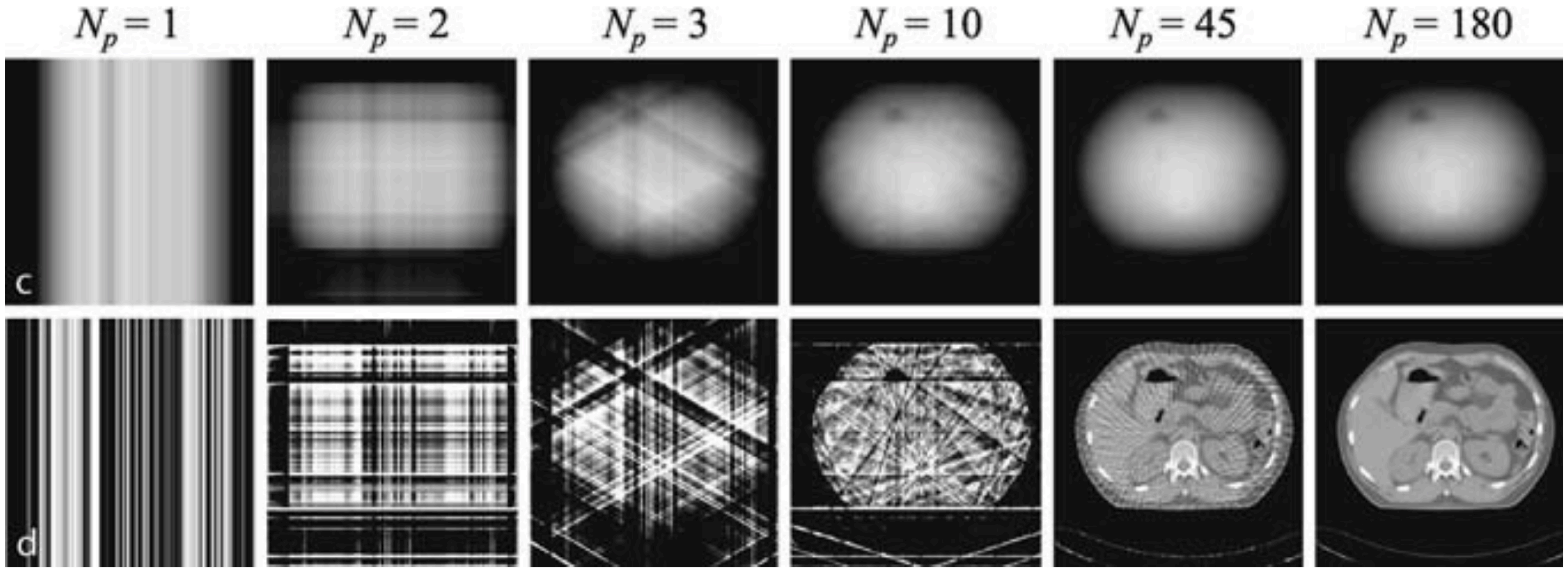


Reconstructed image



Simple vs Filtered Backprojection

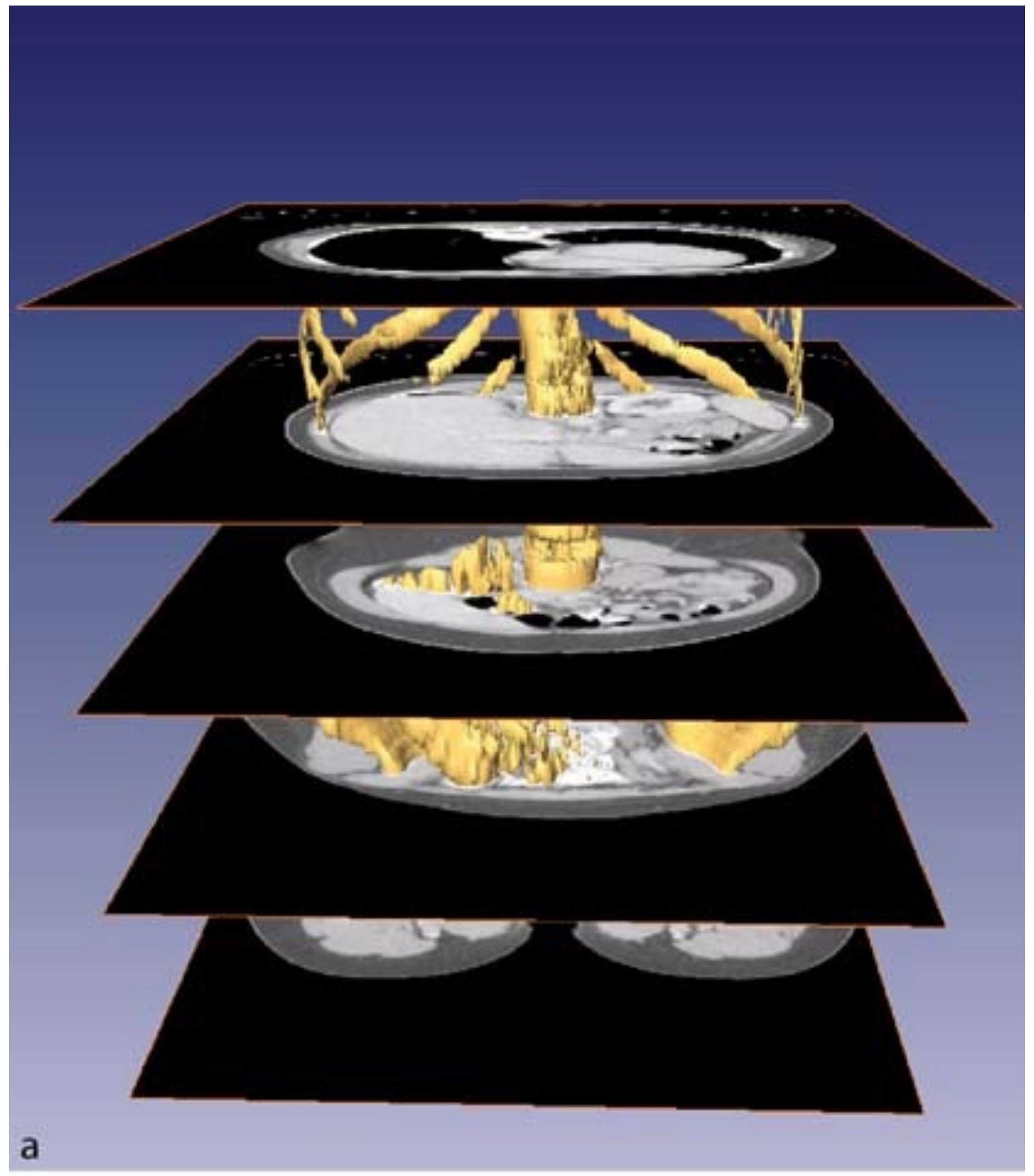
Simple Backprojection

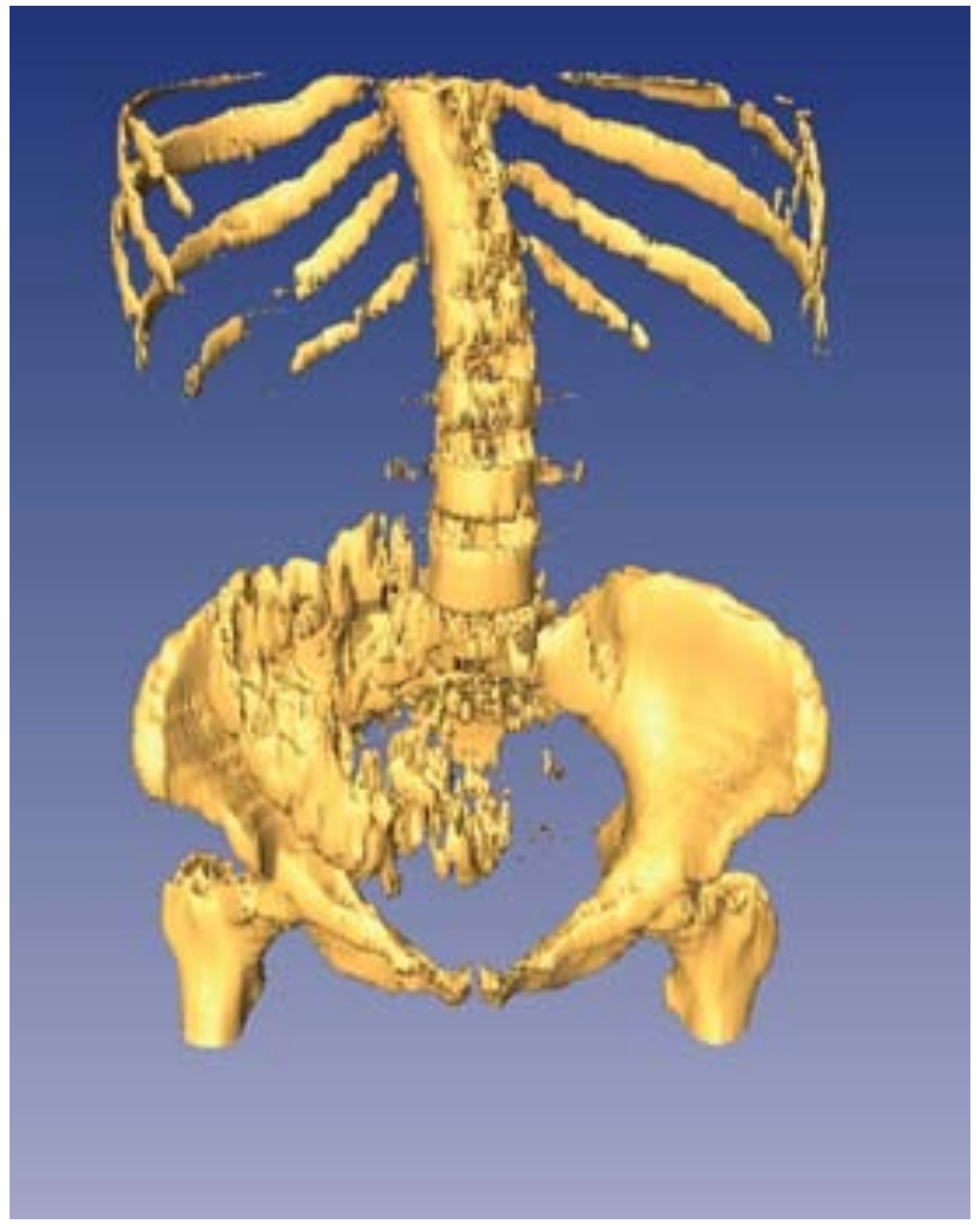


Filtered Backprojection

From 2D to 3D

- **Secondary reconstruction: 3-dimensional representation obtained from stack of 2-dimensional tomographic slices**
- **Requires sophisticated rendering method to visualize data**



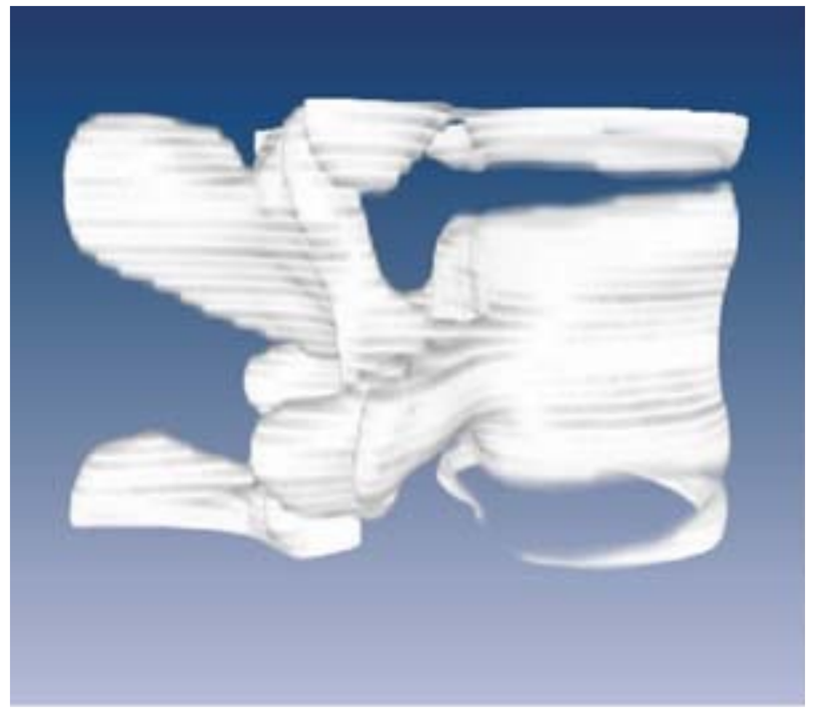


Surface rendering method

Spiral CT (1/2)

- **Problem: Secondary reconstruction from slice stack leads to staircase artifacts**

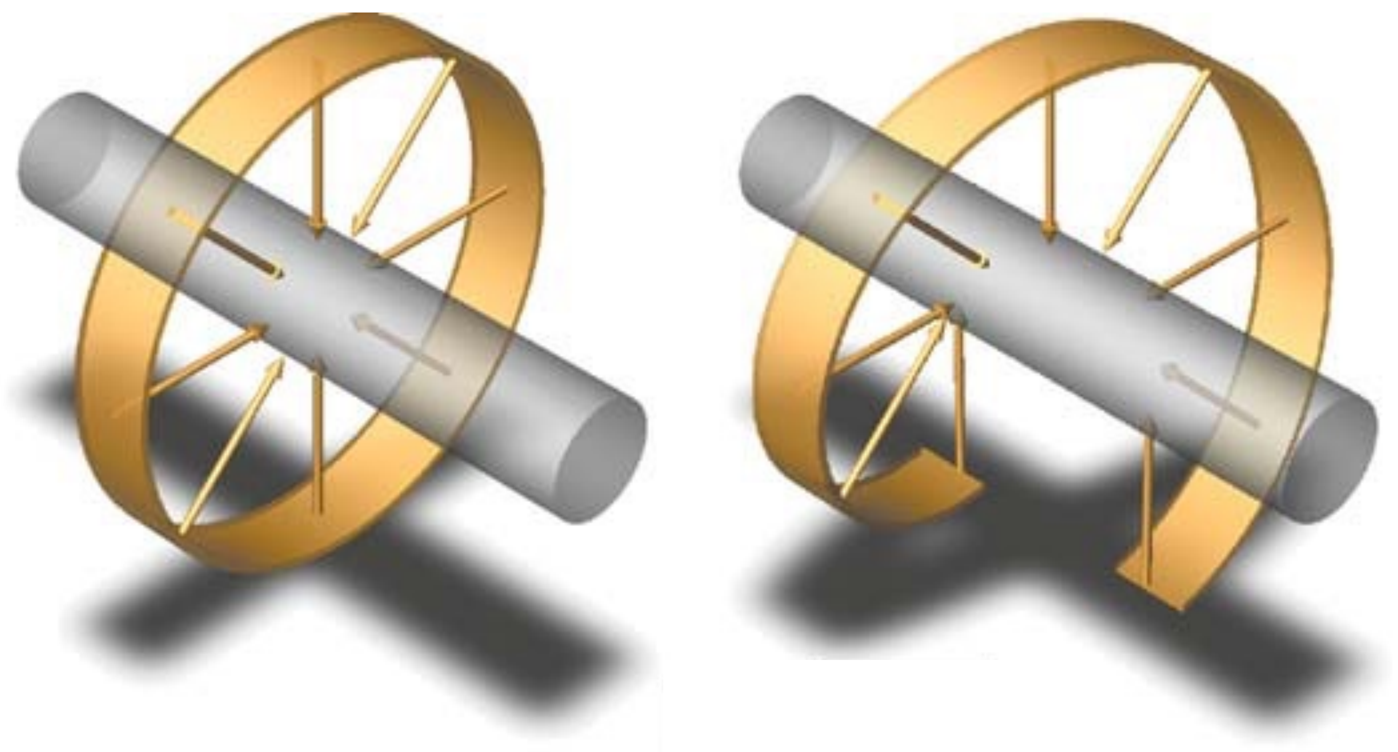
Reconstruction from slice stack

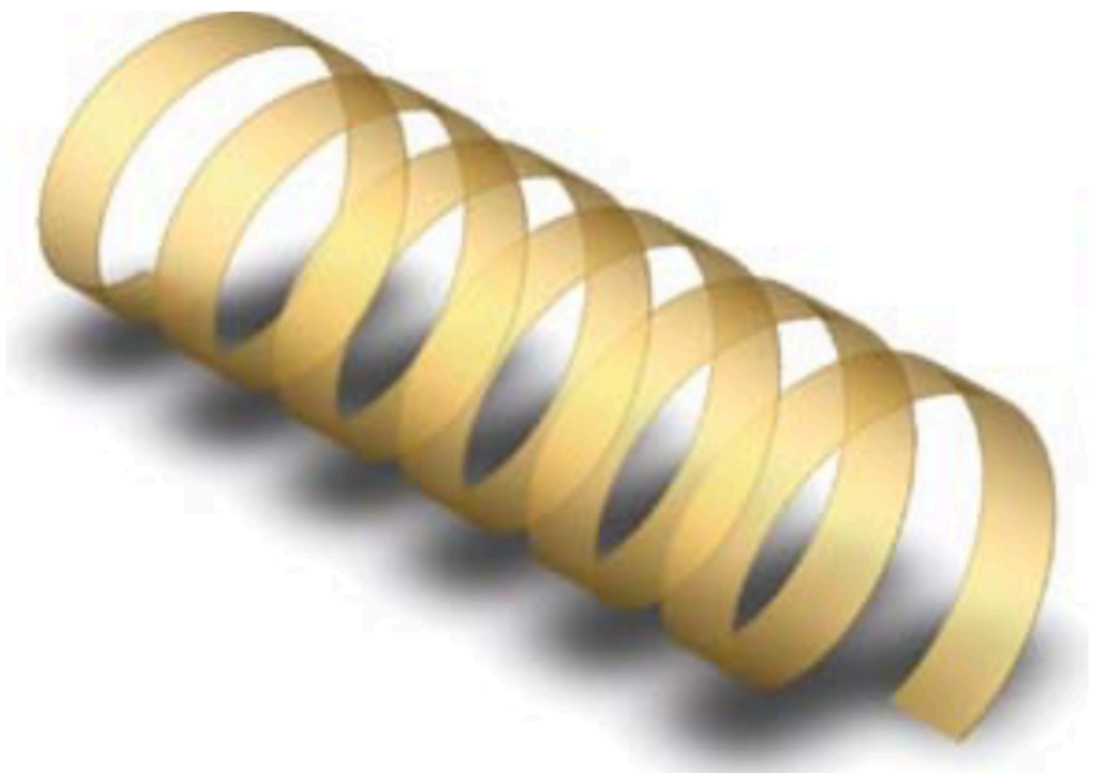


Spiral CT:

- **Constant table feed during data acquisition**
- **Helical trajectory of X-ray source from patient's point of view**
- **Reconstruct missing samples by interpolation**

Spiral CT





Considerably reduced artifacts

Exact 3D Reconstruction in Parallel-Beam Geometry (2/2)

- **Central slice theorem:** $P_{\alpha,\theta}(q,p) = F(q\mathbf{n}_a + p\mathbf{n}_b)$
- **Reconstructing the 3D image: Again 3 easy steps**

Step 1:	$p_{\alpha,\theta}(a,b) \circ\!\!-\!\!\!-\!\!\bullet\ P_{\alpha,\theta}(q,p)$
Step 2:	$P_{\alpha,\theta}(q,p) \bullet\!\!-\!\!\!-\!\!\bullet\ F(u,v,w)$
Step 3:	$F(u,v,w) \bullet\!\!-\!\!\!-\!\!\circ\ f(x,y,z)$

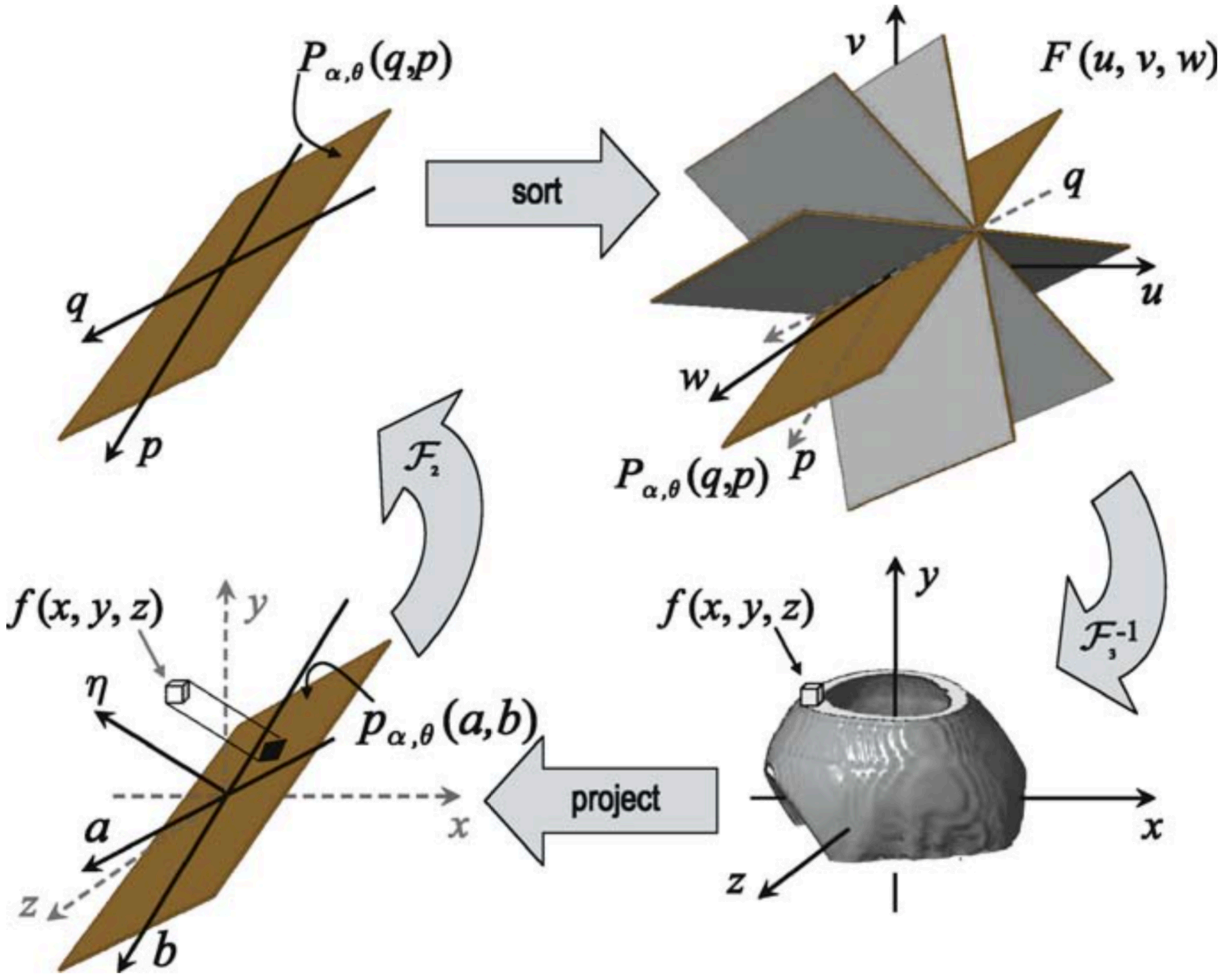


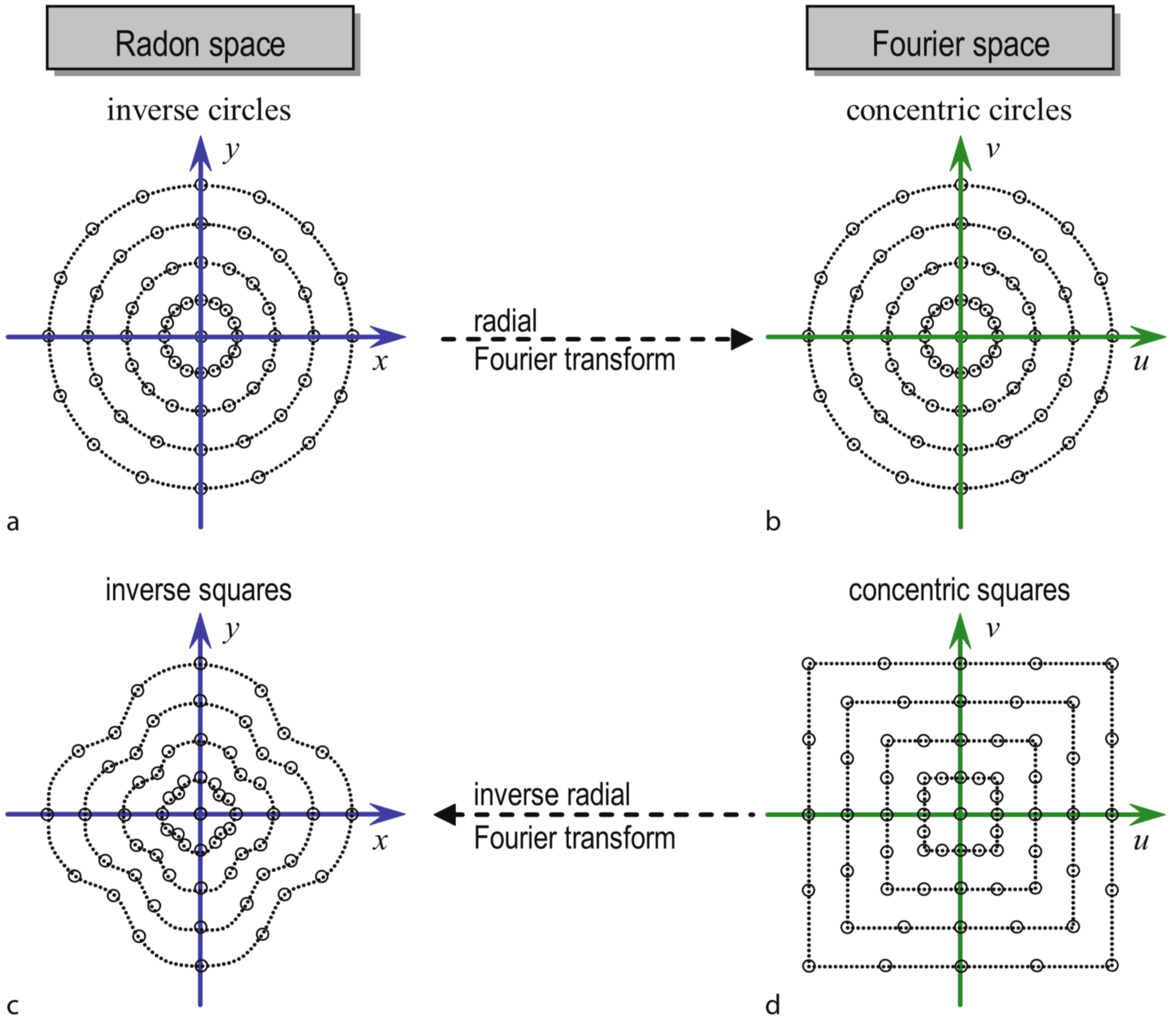
Photo by Owen Beard on Unsplash

Thank you for your attention

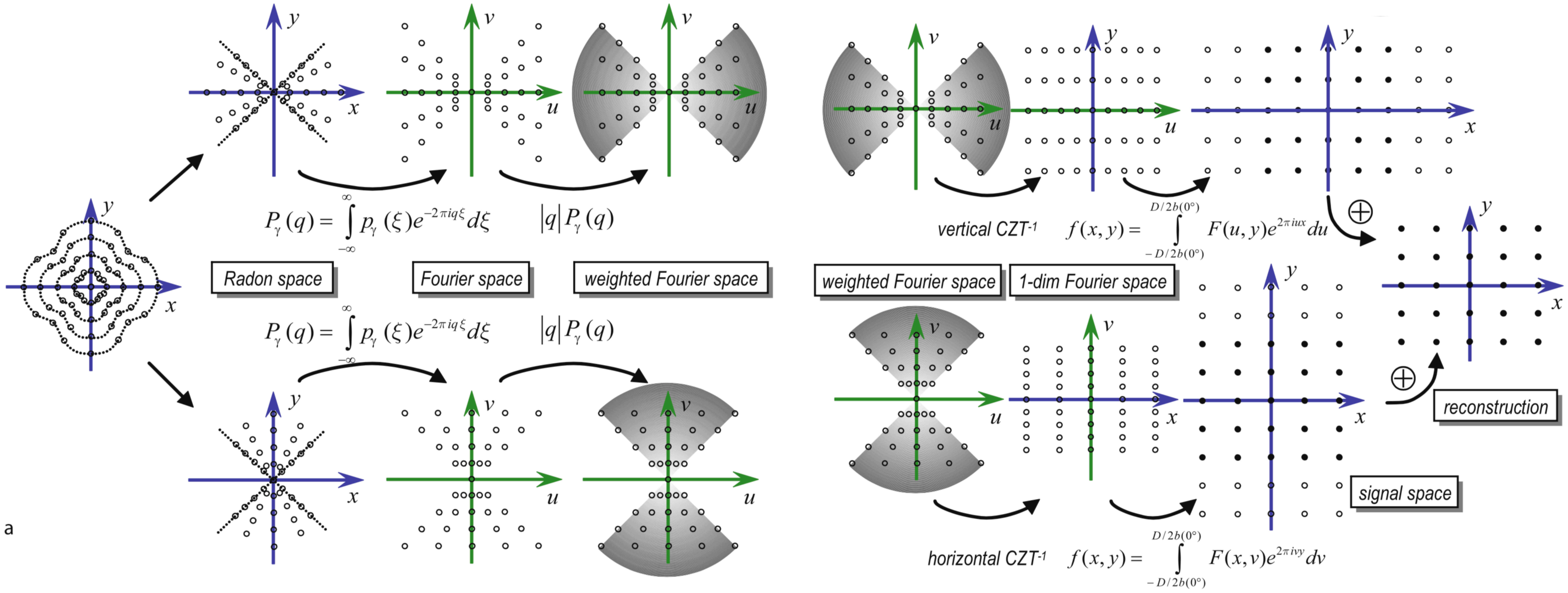
- **Computed Tomography**, Thorsten M. Buzug, *Springer* 2008
- **The Mathematics of Computerized Tomography**, F. Natterer, *SIAM* 1986

Backup-Slides

Linogram-Method



Linogram-Method



Filtered Backprojection

$$f(x,y) = \int_0^{\pi}\int_0^{+\infty} \left(\mathcal{Re}\left(F(q,\gamma)\right) + \mathrm{i}\,\mathcal{Im}\left(F(q,\gamma)\right)\right) \mathrm{e}^{2\pi \mathrm{i} q(x\cos(\gamma)+y\sin(\gamma))} q \,\mathrm{d}q\,\mathrm{d}\gamma$$

$$+ \int_0^{\pi}\int_0^{+\infty} \left(\mathcal{Re}\left(F(q,\gamma)\right) - \mathrm{i}\,\mathcal{Im}\left(F(q,\gamma)\right)\right) \mathrm{e}^{-2\pi \mathrm{i} q(x\cos(\gamma)+y\sin(\gamma))} q \,\mathrm{d}q\,\mathrm{d}\gamma$$

$$= \int_0^{\pi}\int_0^{+\infty} \left(\mathcal{Re}\left(F(q,\gamma)\right) + \mathrm{i}\,\mathcal{Im}\left(F(q,\gamma)\right)\right) \mathrm{e}^{2\pi \mathrm{i} q(x\cos(\gamma)+y\sin(\gamma))} q \,\mathrm{d}q\,\mathrm{d}\gamma$$

$$- \int_0^{\pi}\int_{-\infty}^{0} \left(\mathcal{Re}\left(F(-q,\gamma)\right) - \mathrm{i}\,\mathcal{Im}\left(F(-q,\gamma)\right)\right) \mathrm{e}^{2\pi \mathrm{i} q(x\cos(\gamma)+y\sin(\gamma))} q \,\mathrm{d}q\,\mathrm{d}\gamma \,.$$

$$\mathcal{Re}\{F(q,\gamma)\} \equiv \mathcal{Re}\{F(-q,\gamma+\pi)\} = \mathcal{Re}\{F(-q,\gamma)\} \equiv \mathcal{Re}\{F(q,\gamma+\pi)\}$$

$$\mathcal{Im}\{F(q,\gamma)\} \equiv \mathcal{Im}\{F(-q,\gamma+\pi)\} = -\mathcal{Im}\{F(-q,\gamma)\} \equiv -\mathcal{Im}\{F(q,\gamma+\pi)\}$$

Using the symmetry of F again one obtains

$$f(x,y) = \int_0^{\pi}\int_0^{+\infty} F(q,\gamma)\,\mathrm{e}^{2\pi \mathrm{i} q(x\cos(\gamma)+y\sin(\gamma))} q \,\mathrm{d}q\,\mathrm{d}\gamma$$

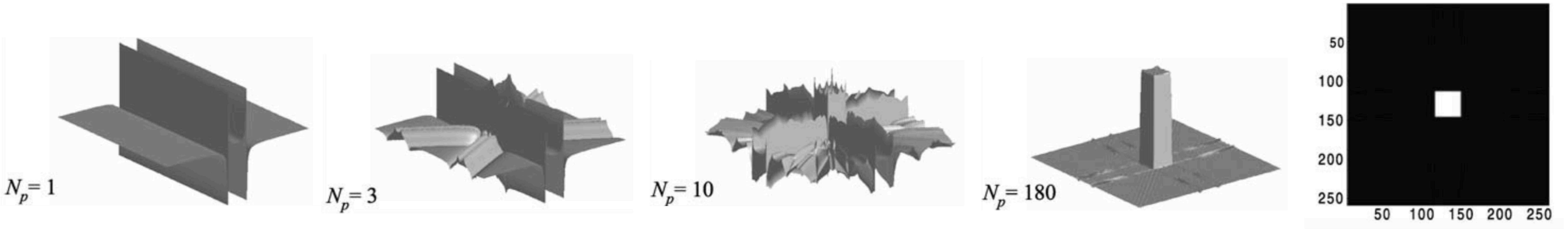
$$+ \int_0^{\pi}\int_{-\infty}^{0} F(q,\gamma)\,\mathrm{e}^{2\pi \mathrm{i} q(x\cos(\gamma)+y\sin(\gamma))} (-q) \,\mathrm{d}q\,\mathrm{d}\gamma \,,$$

which can finally be written as one term

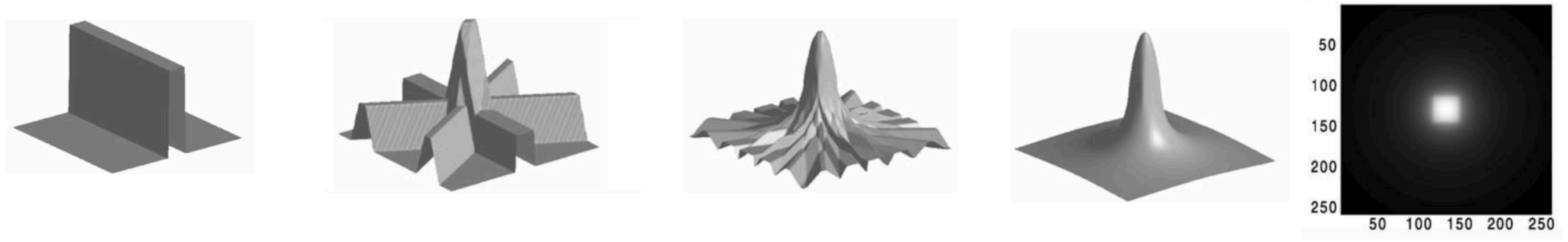
$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{+\infty} F(q,\gamma)\,\mathrm{e}^{2\pi \mathrm{i} q(x\cos(\gamma)+y\sin(\gamma))} |q| \,\mathrm{d}q\,\mathrm{d}\gamma \,.$$

Comparison: Backprojection

Filtered Backprojection



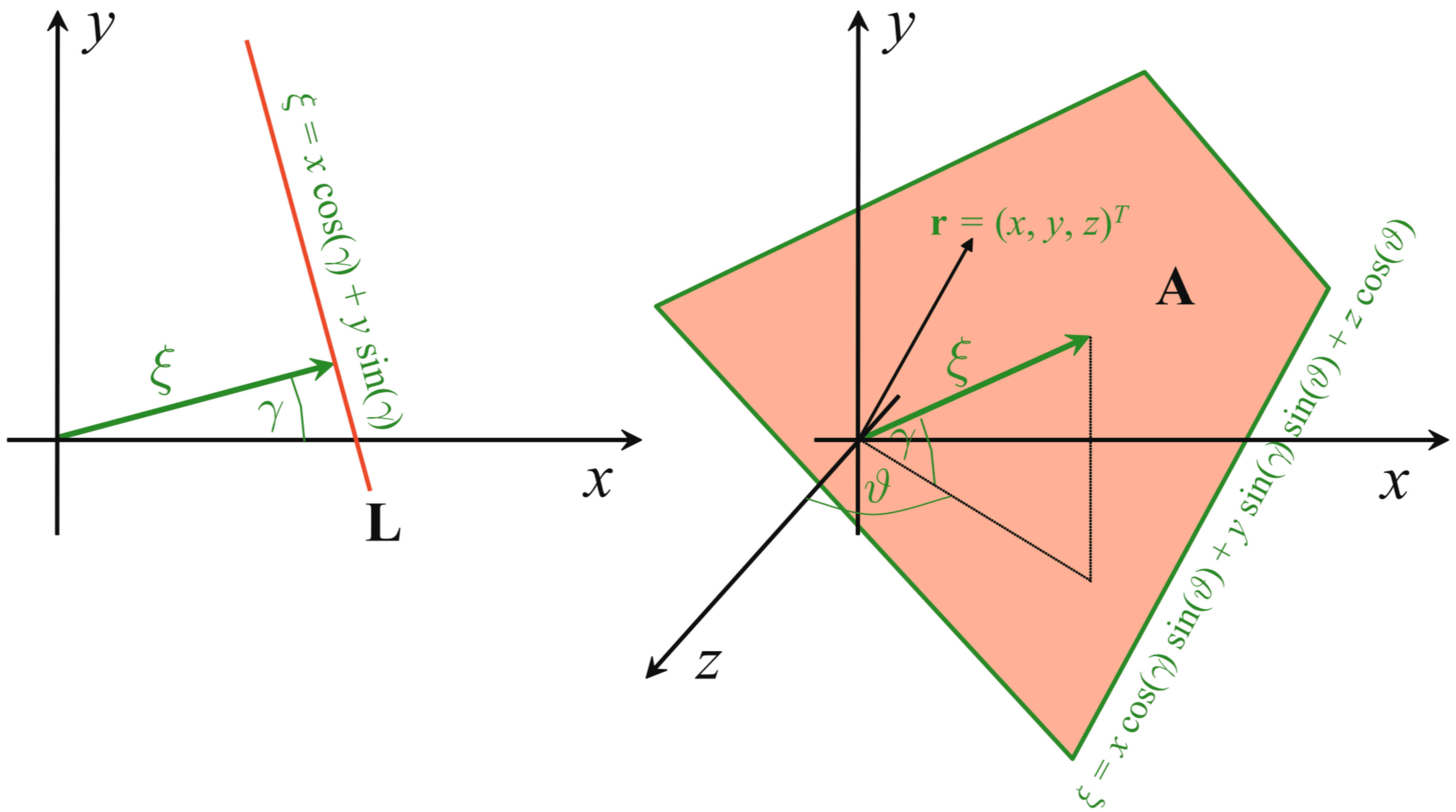
Simple Backprojection



Exact 3D Reconstruction

- **3-dimensional Radon transform:** $f(x,y,z) \overset{\mathscr{R}_3}{\circ\!\!-\!\!\bullet} p_{\gamma,\vartheta}(\xi) = f * \delta(\mathbf{A}) = \int_{\mathbf{r}\in\mathbf{A}} f(\mathbf{r})\mathrm{d}r$
- **Fourier slice theorem:** $F\big(u(q,\gamma,\vartheta), v((q,\gamma,\vartheta), w(q,\gamma,\vartheta)\big) = P_{\gamma,\vartheta}(q)$

Connection to physical measurement?



Exact 3D Reconstruction in Parallel-Beam Geometry (1/2)

- **Hybrid Radon transform:**

$$p_{\alpha,\theta}(a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a'\mathbf{n}_a + b'\mathbf{n}_b + \eta\mathbf{n}_\eta)\cdot\delta(a-a')\delta(b-b')\mathrm{d}a'\mathrm{d}b'\mathrm{d}\eta$$

$$= \int_{-\infty}^{\infty} f(a\mathbf{n}_a + b\mathbf{n}_b + \eta\mathbf{n}_\eta)\mathrm{d}\eta$$

$$P_{\alpha,\theta}(q,p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a\mathbf{n}_a + b\mathbf{n}_b + \eta\mathbf{n}_\eta)e^{-2\pi i(aq+bp)}\mathrm{d}a\mathrm{d}b\mathrm{d}\eta$$

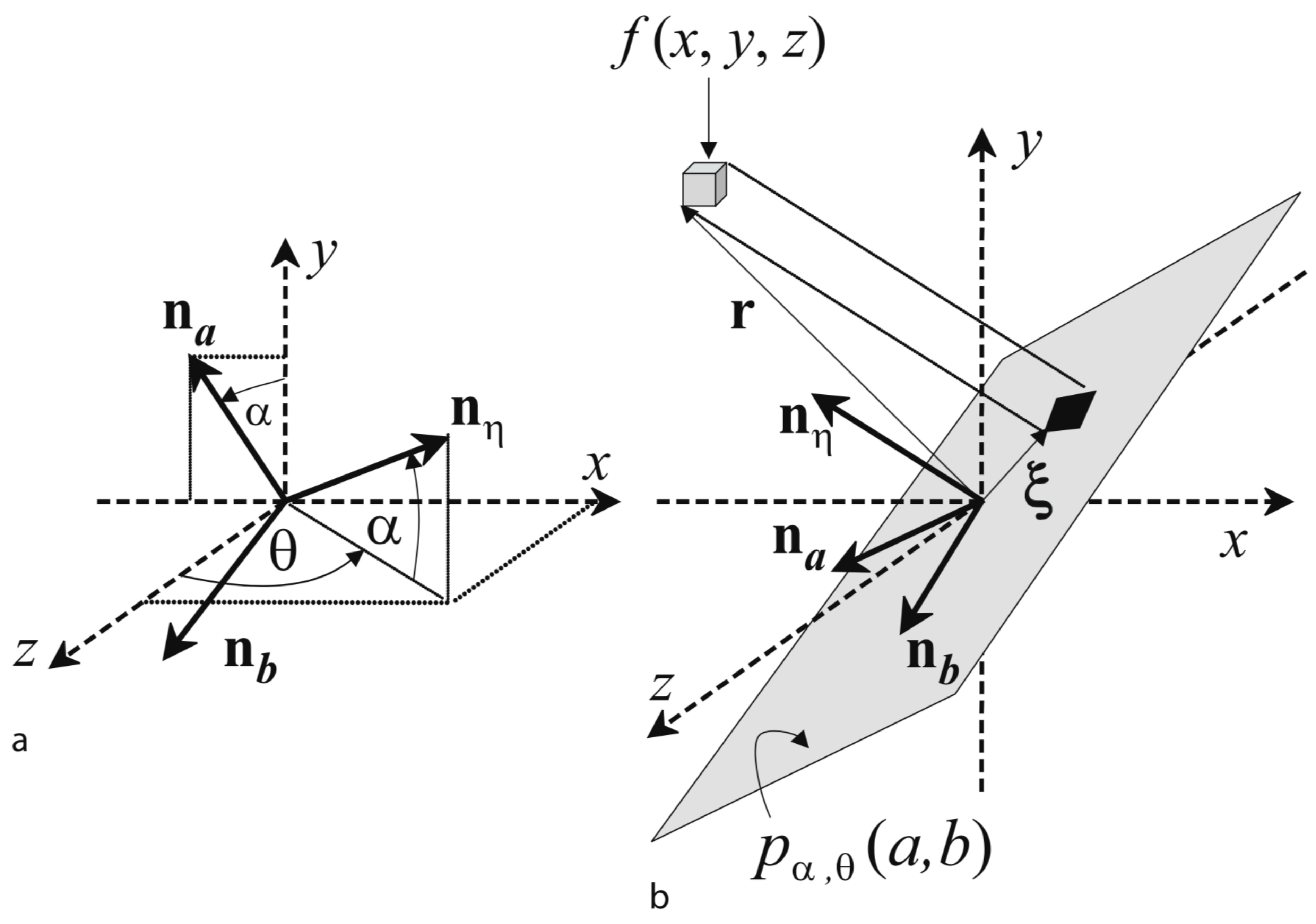
$$(a,b,\eta)^\mathrm{T} \rightarrow (x,y,z)^\mathrm{T}$$

$$P_{\alpha,\theta}(q,p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)e^{-2\pi i\mathbf{r}^\mathrm{T}\cdot(\mathbf{n}_a q + \mathbf{n}_b p)}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

- **Central slice theorem:** $P_{\alpha,\theta}(q,p) = F(q\mathbf{n}_a + p\mathbf{n}_b)$

$$\mathbf{n}_a = \begin{pmatrix}-\sin(\alpha)\\ \cos(\alpha)\\ 0\end{pmatrix} \quad \mathbf{n}_b = \begin{pmatrix}-\cos(\alpha)\cos(\theta)\\ -\sin(\alpha)\cos(\theta)\\ \sin(\theta)\end{pmatrix} \quad \mathbf{n}_\eta = \begin{pmatrix}\cos(\alpha)\sin(\theta)\\ \sin(\alpha)\sin(\theta)\\ \cos(\theta)\end{pmatrix}$$

$$f(\mathbf{r}) = f(x,y,z) = f(a\mathbf{n}_a + b\mathbf{n}_b + \eta\mathbf{n}_\eta)$$



Cone-Beam Geometry